EXAMPLES IN
ALGEBRA
FOR
SENIOR CLASSES.

3/-

EXAMPLES IN ALGEBRA,

FOR SENIOR CLASSES.

COMPRISING

NUMEROUS GRADUATED EXAMPLES IN

FRACTIONS, SURDS, EQUATIONS, PROGRESSIONS, &c.

WITH THE

EXAMINATION PAPERS FOR CIVIL SERVICE, STAFF, AND ARTILLERY APPOINTMENTS; COLLEGE OF PRECEPTORS, LONDON UNIVERSITY, AND OXFORD AND CAMBRIDGE MIDDLE CLASS EXAMINATIONS.

BY

J. WHARTON, M.A., M.C.P.,

LATE EXAMINER IN MATHEMATICS FOR THE COLLEGE OF PRECEPTORS.

Second Edition.

LONDON:

C. F. HODGSON, 1, GOUGH SQUARE,

FLEET STREET.

1860.

LONDON:
PRINTED BY C. F. HODGSON,
GOUGH SQUARE, FLEET STREET.

PREFACE.

In order to extend the study of the higher parts of Algebra, I have attempted to produce a series of Graduated Examples suitable for Senior Classes. For this purpose I have made large selections from my former Examples, invented new ones, and selected many from recent Examination Papers; in order that the Pupil, having a number of Examples of one kind before him, may become perfectly master of any peculiarities or artifices before he leave them. I have also introducccd Examples in every department which the Oxford, Cambridge, Civil Service, or Military Examinations have touched upon.

Complete Solutions of all the Questions in this work are in course of publication.

42, Queen Square, W.C.
Oct. 15*th*, 1859.

CONTENTS.

EXAMPLES IN ALGEBRA.

SUBSTITUTIONS.

I. (1.) Find the value of $\dfrac{x^2 - ax + a^2}{x^2 + ax + a^2}$; when $x = \frac{1}{2}$ and $a = \frac{1}{4}$.

By substituting for x and a the values, $\frac{1}{2}$ and $\frac{1}{4}$, we have

$$\frac{x^2 - ax + a^2}{x^2 + ax + a^2} = \frac{\frac{1}{4} - \frac{1}{8} + \frac{1}{16}}{\frac{1}{4} + \frac{1}{8} + \frac{1}{16}} = \frac{4 - 2 + 1}{4 + 2 + 1} = \frac{3}{7}.$$

(2.) $\dfrac{x - y}{x + y} + \dfrac{x^2 - y^2}{x^2 + y^2}$; when $x = \frac{3}{2}$, $y = \frac{2}{3}$.

(3.) $\dfrac{x^2 - 2xy + y^2}{x^4 + x^2y^2 + y^4}$; when $x = \frac{1}{3}$, $y = \frac{1}{4}$.

(4.) $\sqrt[3]{x^3 - 3x^2y + 3xy^2 - y^3} \times \sqrt{x^2 - 2xy + y^2}$; when $x = \frac{3}{4}$, $y = \frac{4}{5}$.

(5.) $4a^4 - 11a^3 + 12a^2 - 4$; when $a = 4$.

(6.) $\dfrac{a - b}{a + b} + \sqrt{\dfrac{a^2 + b^2}{a^2 - b^2}}$; when $a = \frac{1}{4}$, $b = \frac{1}{5}$.

(7.) $(x^3 - 3x^2 + 2x) \cdot (x^3 + 3x^2 - 2x)$; when $x = \frac{4}{5}$.

(8.) $x^2 - 16 + \sqrt{x^2 - 16} - 12$; when $x = 4\sqrt{2}$.

(9.) $x^3 - 2x^2 + x - 2$; when $x = 2 \pm \sqrt{-5}$.

(10.) $\sqrt{12x^2 - 84} - x^2 + \sqrt{x^2 - 7}$; when $x = 2 - \sqrt{3}$.

(11.) $\dfrac{a}{b} - \sqrt{\dfrac{1 + a}{1 - b}}$; when $a = \frac{1}{4}$, $b = \frac{1}{5}$.

(12.) $\dfrac{x^2 + 2xy + y^2}{x^2 - 2xy + y^2}$; when $x = \frac{1}{3}$, $y = \frac{2}{3}$.

(13.) $\sqrt{\dfrac{1 + x}{1 - y}} + \sqrt{\dfrac{3\,(1 - 2x^2)}{1 - y^2}} + \sqrt{x^2 - 4xy + 4y^2}$;
when $a = \frac{1}{4}$, $b = \frac{1}{3}$.

EQUALITIES AND INEQUALITIES.

II. (1.) Which is the greater, $2^{\frac{1}{2}}$ or $3^{\frac{1}{3}}$?

$$2^{\frac{1}{2}} > \text{ or } < \text{ (greater or less than) } 3^{\frac{1}{3}};$$

raise both sides to the sixth power;

$$\text{then } 2^3 > \text{ or } < 3^2;$$
$$\text{but } 3^2 = 9, \text{ and } 2^3 = 8;$$
$$\therefore\ 3^2 > 2^3; \text{ and consequently } 3^{\frac{1}{3}} > 2^{\frac{1}{2}}.$$

(2.) Which is greater, $\dfrac{19}{20}$ or $\dfrac{18}{19}$?

$$\frac{19}{20} > \text{ or } < \frac{18}{19},\ \therefore\ 361 > \text{ or } < 360;$$
$$\text{but } 361 > 360,\ \therefore\ \frac{19}{20} > \frac{18}{19}.$$

(3.) Show that $\dfrac{a}{b^2} + \dfrac{b}{a^2} > \dfrac{1}{a} + \dfrac{1}{b}$.

Multiplying both sides by a^2b^2,

$$a^3 + b^3 > \text{ or } < ab^2 + a^2b > \text{ or } < ab\,(a + b).$$

Dividing both sides by $a + b$,

$$a^2 - ab + b^2 > \text{ or } < ab;$$
$$\text{then } a^2 - 2ab + b^2, \text{ or } (a - b)^2, > 0,$$

since all rational quantities when squared are positive, and therefore greater than *nothing*;

$$\therefore \frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{a} + \frac{1}{b}$$

(4.) Show that $6^{\frac{1}{2}} > 12^{\frac{1}{3}}$; that $\frac{19}{20} < \frac{20}{21}$; and that $\sqrt{19} + \sqrt{3} > \sqrt{10} + \sqrt{7}$.

(5.) Show that $a^6 + a^4b^2 + a^2b^4 + b^6 > (a^3 + b^3)^2$.

(6.) Show that $n^3 + 1 > n + n^2$.

(7.) Show that $3(1+a^2+a^4) > (1 + a + a^2)^2$; and that $a^m - b^m < ma^{m-1}(a - b)$ and $> mb^{m-1}(a - b)$, if a be $> b$.

(8.) Show that $xy > ac + bd$, if $x^2 = a^2 + b^2$ and $y^2 = c^2 + d^2$.

(9.) Show that $2(1 + x^2 + x^4) > 3(1 + x^3)$.

(10.) Show that $abc > (a + b - c)(a + c - b)(b + c - a)$, unless a, b, and c are equal.

(11.) Show that $(a + b + c)^2 < 3(a^2 + b^2 + c^2)$, unless $a = b = c$.

(12.) Show that $\sqrt[3]{a} > \sqrt[4]{a + 1}$, if a be not < 3.

(13.) Show that

$$\frac{2a\sqrt{1 + x^2}}{x + \sqrt{1 + x^2}} = a + b, \text{ if } x = \tfrac{1}{2}\sqrt{\frac{a}{b}} - \tfrac{1}{2}\sqrt{\frac{b}{a}}.$$

(14.) Show that $a(b + c)^2 + b(a + c)^2 + c(a + b)^2 - (a + b)(a - c)(b - c) - (a - b)(a - c)(b + c) + (a - b)(b - c)(a + c) = 12abc$.

(15.) Show that $(a^2 - b^2)c + (b^2 - c^2)a + (c^2 - a^2)b = (a - b)(b - c)(c - a)$.

(16.) Show that

$$\left(x + \frac{1}{x}\right)^2 - \left(y + \frac{1}{y}\right)^2 = \left(xy - \frac{1}{xy}\right) \times \left(\frac{x}{y} - \frac{y}{x}\right),$$

and prove it when $x = \frac{1}{2}$ and $y = \frac{1}{4}$.

(17.) If $(a^2 + bc)^2(b^2 + ac)^2(c^2 + ab)^2 = (a^2 - bc)^2(b^2 - ac)^2(c^2 - ab)^2$, show that either

$$a^3 + b^3 + c^3 + abc = 0, \text{ or } \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{abc} = 0.$$

(18.) Show that $a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2) = [a^2(b-c) + b^2(c-a) + c^2(a-b)] \times (a+b)(b+c)(a+c)$.

(19.) Prove that $(a + b + c + d)(a + b - c - d)(a-b-c+d)(a-b+c-d) + (b-a+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) = 16abcd$.

G. C. M. & L. C. M.

III. Find the Greatest Common Measure of—

(1.) $6a^2 + 7ax - 3x^2$ and $6a^2 + 11ax + 3x^2$.

(2.) $x^4 + a^2x^2 + a^4$ and $x^4 + ax^3 - a^3x - a^4$.

(3.) $6x^4 - 25a^2x^2 - 9a^4$ and $3x^3 - 15ax^2 + a^2x - 5a^3$.

(4.) $x^3 - 19x^2 + 119x - 245$ and $3x^2 - 38x + 119$.

(5.) $3x^3 - 22x - 15$ and $5x^4 - 17x^3 + 18x$.

(6.) $x^3 - 3x^2 + 7x - 21$ and $2x^4 + 19x^2 + 35$.

(7.) $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$.

(8.) $a^4 - x^4$ and $a^4 + a^3x - ax^3 - x^4$.

(9.) $x^6 + x^2y - x^4y^2 - y^3$ and $x^4 - x^2y - x^2y^2 + y^3$.

(10.) $6a^4x^3 - 10a^2x^4y - 9a^3x^2y^2 + 15ax^3y^3$ and $10a^4xy^2 - 15a^3y^4 + 8a^2x^2y^3 - 12axy^5$.

(11.) $x^5 - x^4 - x + 1$ and $5x^4 - 4x^3 - 1$.

(12.) $x^3 - 8x + 3$ and $x^6 + 3x^5 + x + 3$.

(13.) $x^3 + xy^2 + x^2y + y^3$ and $x^4 - y^4$.

(14.) $x^4 + ax^3 - 9a^2x^2 + 11a^3x - 4a^4$ and $x^4 - ax^3 - 3a^2x^2 + 5a^3x - 2a^4$.

Find the Least Common Multiple of—

(15.) $a^3 + x^3$ and $a^2 - x^2$.

(16.) $2x - 1$, $4x^2 - 1$, and $4x^2 + 1$.

(17.) $x - 1$, $x - 2$, $x^2 - 4$, and $x + 1$.

(18.) $4(1 - x)^2$, $8(1 - x)$, $8(1 + x)$, and $4(1 + x^2)$.

(19.) $a^3 - x^3$, $a^2 - x^2$, and $a^3 + x^3$.

(20.) $x - a$, $x + a$, $x^2 - a^2$, and $x^2 + a^2$.

(21.) $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ and $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$.

(22.) $a^2 - b^2$, $a^2 + b^2$, $(a - b)^2$, $(a + b)^2$, $a^3 - b^3$ and $a^3 + b^3$.

(23.) $x^3 - 3x^2 + 3x - 1$, $x^3 - x^2 - x + 1$, $x^4 - 2x^3 + 2x - 1$, $x^4 - 2x^3 + 2x^2 - 2x + 1$.

INVOLUTION AND EVOLUTION.

IV. Find the square of—

(1.) $2a + 3b$, $\frac{a}{2} \pm \frac{b}{3}$, and $a - 3b$.

(2.) $1 + x + x^2$, $1 + \frac{x}{2} + \frac{x^2}{4}$, and $\frac{x^2}{4} + \frac{xy}{3} + \frac{y^2}{4}$.

(3.) $x - 1 + \frac{1}{x}$, $x + 1 - \frac{1}{x}$, and $\frac{x^2}{a^2} - \frac{xy}{ab} + \frac{y^2}{b^2}$.

Find the cube of—

(4.) $a + b$, $a + 2b$, and $\frac{a}{2} + \frac{b}{3}$.

(5.) $x^2 + ab + a^2$, and $1 - \frac{x}{2} + \frac{x^2}{4}$.

(6.) $1 + 2x + 3x^2$, $x^{\frac{1}{2}} - y^{\frac{1}{3}}$, and $x^{\frac{1}{3}} - 2y^{\frac{1}{3}}$.

(7.) $x - x^{-1}$, $x - 1 + x^{-1}$, and $x^{\frac{1}{3}} + 1 - x^{-\frac{1}{3}}$.

Find the square root of

(8.) $9a^4b^2c^2$, $64a^6b^4c^6$, and $576a^{2m}b^{2n}c^{4n}$.

(9.) $16a^2 - 40ab + 25b^2$, and $16a^4 - 48a^3 + 100a^2 - 96a + 64$,

(10.) $a^2 - 2ab + c^2 + b^2 + 2ac - 2bc$.

(11.) $x^6 - 6x^5y + 16x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.

(12.) $x^4 - \frac{x}{2} + \frac{3x^2}{2} + \frac{1}{16} - 2x^3$.

(13.) $\frac{a^4}{9} + \frac{2a^3x}{3} + ax^3 + \frac{x^4}{4} + \frac{4a^2x^2}{3}$,

(14.) $x^2 + \frac{1}{x^2} + 2\left(x - \frac{1}{x}\right) - 1$.

(15.) $25\frac{8}{7} - \frac{20x}{7y} + \frac{9y^2}{16x^2} - \frac{15y}{2x} + \frac{4x^2}{49y^2}$.

(16.) $4x^4 - 8x^3 + 6x^2 - 2x + \frac{1}{4}$, and $\frac{a^2}{b^2} + \frac{2a}{b} + \frac{2b}{a} + 3 + \frac{b^2}{a^2}$.

(17.) $\frac{x^2}{4} - \frac{bx}{4} - \frac{ab}{12} + \frac{ax}{6} + \frac{b^2}{16} + \frac{a^2}{36}$.

(18.) $\frac{1051x^2}{25} - \frac{6x}{5} - \frac{14x^3}{5} + 9 + 49x^4.$

(19.) $\frac{x^2}{y^2} + \frac{y^2}{x^2} - \sqrt{2}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{5}{2}.$

(20.) $a^2 + x.$ (21.) $\frac{a^4}{a^2 - x}.$ (22.) $\frac{a + x}{a - x}.$

Find the cube root of—

(23.) $x^6 - 3a^4x^4 + 3a^8x^2 - a^{12}.$

(24.) $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$

(25.) $\frac{a^3}{8} + \frac{8}{27a^6} + \frac{2}{3a^3} + \frac{1}{2}.$

(26.) $\frac{x^3}{y^6} - 3\frac{x}{y^2} - \frac{y^6}{x^3} + 3\frac{y^2}{x}.$

(27.) $x^3 - 12x^2 + 54x - 112 + \frac{108}{x} - \frac{48}{x^2} + \frac{8}{x^3}$

(28.) $\frac{a^3c^3}{b^3} - \frac{3a^2c}{bx} + \frac{3ab}{cx^2} - \frac{b^3}{c^3x^3}.$

(29.) $e^{3x} - e^{-3x} - 3(e^x - e^{-x})$ and $\frac{x^6}{a^6} - \frac{3x^2}{a^2} + \frac{3a^2}{x^2} - \frac{a^6}{x^6}.$

(30.) $x^3 - \frac{1}{x^3} + 3\left(x - \frac{1}{x}\right)$ and $x - \frac{1}{x^3} - 3x^2 - \frac{3}{x^2} + 5.$

(31.) Find the 4th root of

$$16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$$

(32.) Find the 4th root of

$$x^8 - \frac{3a^2x^7}{b} + \frac{27a^4x^6}{8b^2} - \frac{27a^6x^5}{16b^2} + \frac{81a^8x^4}{256b^4}.$$

(33.) Find the square root of

$$(a - b)^2 [(a - b)^2 - 2(a^2 + b^2)] + 2(a^4 + b^4).$$

(34.) Find the square root of $m^2n^2 + m^2r^2 + n^2r^2$,

when $m = \frac{1}{ab^2} + \frac{1}{a^2b}$, $n = \frac{2}{ab^2} + \frac{1}{a^2b}$, $r = \frac{3}{ab^2} + \frac{2}{a^2b}.$

(35.) Find the 6th root of

$$x^6 + \frac{1}{x^6} - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 - \frac{1}{x^2}\right) - 20.$$

(36.) Find the 5th root of

$$32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1.$$

FRACTIONS.

V. Find the sum of—

(1.) $\frac{b}{12a} + \frac{3b}{4a} + \frac{5b}{8a} - \frac{13b}{16a}$.

(2.) $\frac{3}{2x} + \frac{5}{4x} + \frac{6}{15x} - \frac{11}{20x}$.

(3.) $\frac{4x - 30}{15a} - \frac{3x - 15}{5a} + \frac{10x - 11}{30a}$.

(4.) $\frac{4x - 3y}{3(1 - y)} - \frac{x + 3y}{3(1 - y)} + \frac{2y}{1 - y} - \frac{x}{1 - y}$.

(5.) $\frac{3m - 4n}{7} - \frac{2m - n - 1}{3} + \frac{15m - 4}{12} - \frac{85m - 20n}{84}$.

(6.) $\frac{x^4 - a^4}{(x - a)^2} \div \left(\frac{x^2 + ax}{x - a}\right) \times \left(x + \frac{a^2}{x}\right)$

(7.) $\frac{a}{c} - \frac{(ad - bc)x}{c(c + dx)} - \frac{a}{c + dx} - \frac{bx}{c + dx}$.

(8.) $\frac{1}{2} \times \frac{3x + 2y}{3x - 2y} - \frac{1}{2} \times \frac{3x - 2y}{3x + 2y}$.

(9.) $\frac{x^2 + xy}{x - y} \times \frac{(x - y)^2}{x^4 - y^4} \div \frac{x}{x^2 + y^2}$.

(10.) $\left(\frac{1}{1 + x} + \frac{x}{1 - x}\right) \div \left(\frac{1}{1 - x} - \frac{x}{1 + x}\right)$.

(11.) $\frac{a - b}{a + b} + \frac{2ab}{a^2 - b^2} - \frac{a^2 + b^2}{a^2 - b^2}$.

(12.) $\frac{b}{a - b} + \frac{a}{a + b} - \frac{a}{a - b} + \frac{b}{a + b}$.

(13.) $\frac{1}{x - 1} - \frac{1}{2x + 2} - \frac{x + 3}{2x^2 + 2} - \frac{x + 3}{x^4 - 1}$.

(14.) $\frac{1}{2(x - 1)} - \frac{1}{2(x + 1)} - \frac{1}{x^2} - \frac{1}{x^2(x^2 - 1)}$.

(15.) $\frac{x + y}{y} - \frac{2x}{x + y} + \frac{x^3 - x^2y}{y^3 - x^2y}$.

(16.) $\frac{x}{1-x} - \frac{x^2}{(1-x)^2} + \frac{x^3}{(1-x)^3} - x.$

(17.) $\frac{1}{4(1+x)} + \frac{1}{4(1-x)} + \frac{1}{2(1+x^2)}.$

(18.) $\frac{3}{4(1-x)^2} + \frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)}.$

(19.) $\frac{a+c}{(a-b)(x-a)} - \frac{b+c}{(a-b)(x-b)} - \frac{x+c}{(x-a)(x-b)}.$

(20.) $\frac{1}{(x+1)(x+2)} - \frac{1}{(x+1)(x+2)(x+3)} - \frac{1}{(x+1)(x+3)}.$

(21.) $\frac{1}{(x+1)(x+2)} - \frac{3}{(x+1)(x+2)(x+3)} - \frac{x}{(x+1)(x+2)(x+3)}$

(22.) $\frac{1}{1+x} + \left(1 - \frac{1}{1+x}\right) - \frac{1}{x}.$

(23.) $(a-x)^2 + \frac{6a^2x + 2x^3}{a-x} - \frac{(a+x)^3}{a-x}.$

(24.) $1 - \frac{a^2+b^2-c^2}{2ab} - \frac{(a+c-b)(b+c-a)}{2ab}.$

(25.) $b^2 - \frac{(b^2+c^2-a^2)^2}{4c^2} - \frac{(a+b+c)(a+c-b)(a+b-c)(b+c-a)}{4c^2}.$

(26.) $1 - \frac{a^2+b^2-c^2-d^2}{2(ab+cd)} - \frac{(c+d)^2-(a-b)^2}{2(ab+cd)}.$

(27.) $\left(\frac{y}{1+y} + \frac{1-y}{y}\right) + \left(\frac{y}{1+y} - \frac{1-y}{y}\right).$

(28.) $\frac{a+2x}{a-x} \times \frac{(a-x)^2}{(a+2x)^2} - \frac{a-x}{a+2x}.$

(29.) $\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3} \times \frac{a^2-ax+x^2}{a+x} \div \frac{a^4+a^2x^2+x^4}{a^4-x^4}.$

(30.) $\frac{pr+(pq+qr)x+q^2x^2}{p-qx} \times \frac{ps+(pt-qs)x-qtx^2}{p+qx}.$

(31.) $\frac{a^2+(2ac-b^2)x^2+c^2x^4}{a^2+2abx+(2ac+b^2)x^2+2bcx^3+c^2x^4} \div \frac{a^2+(ac-b^2)x^2+bcx^3}{a^2+(ac-b^2)x^2-bcx^3}.$

(32.) $\frac{1}{2(1-x+x^2)} + \frac{1}{2(1+x+x^2)} - \frac{x^2+1}{1+x^2+x^4}.$

(33.) $\dfrac{1-x}{1+x}+\dfrac{(1-x)(1-x^2)}{(1+x)(1+x^2)}-\dfrac{2(1-x)}{(1+x)(1+x^2)}.$

(34.) $\dfrac{x}{x-3}-\dfrac{x^2-9}{x(x+3)}+\dfrac{x}{x+3}-\dfrac{x+3}{x}.$

(35.) $\dfrac{(x+1)}{2(x^2-1)}+\dfrac{9}{2(x-3)}-\dfrac{4}{x-2}.$

(36.) $\dfrac{a-b}{ab}+\dfrac{c-a}{ac}+\dfrac{b-c}{bc}.$

(37.) $\dfrac{2a+b}{a-b}-\dfrac{a-2b}{a+b}-\dfrac{a^2-3ab+b^2}{a^2+b^2}.$

(38.) $\dfrac{a^3}{(a+b)^3}-\dfrac{ab}{(a+b)^2}+\dfrac{b}{a+b}.$

(39.) $\dfrac{x(x-2y)^3}{(x+y)^3}+\dfrac{y(2x-y)^3}{(x+y)^3}.$

(40.) $\dfrac{(1+x)^2}{(3-x)^2}+\dfrac{1-x^2}{(3-x)^2}-\dfrac{1+x^2}{(3-x)^2}.$

(41.) $\dfrac{x-a}{x+a}+\dfrac{x+a}{x-a}-\dfrac{4ax}{x^2-a^2}.$

(42.) $\dfrac{1-2x}{3(x^2-x+1)}+\dfrac{1+x}{2(x^2+1)}+\dfrac{1}{6(x+1)}.$

(43.) $\dfrac{12}{5(x+3)}-\dfrac{1}{15(x-2)}-\dfrac{4}{3(x+1)}.$

(44.) $\dfrac{3}{4(1-x^2)}+\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}-\dfrac{1-x}{4(1+x^2)}.$

(45.) $\dfrac{1}{2(x+1)}-\dfrac{4}{x+2}+\dfrac{9}{2(x+3)}.$

(46.) $\dfrac{1}{1+x}+\dfrac{1}{1-x}+\dfrac{2}{1+2x}+\dfrac{2}{1-2x}.$

(47.) $\dfrac{3}{(1+x)^3}-\dfrac{1}{1+x}-\dfrac{1}{1-x}+\dfrac{2x}{(1-x)(1+x)^2}.$

(48.) $\dfrac{1}{5(x-1)}-\dfrac{1}{4x^2}-\dfrac{3}{16x}-\dfrac{1}{80(x+4)}.$

(49.) $\dfrac{1}{(1+x)^2}+\dfrac{3}{1+x}+\dfrac{5}{(1+2x)^2}-\dfrac{6}{1+2x}.$

(50.) $\frac{1}{x^3} + \frac{1}{x^2} + \frac{2}{x} + \frac{1}{2(1-x)^2} + \frac{7}{4(1-x)} - \frac{1}{4(1+x)}$.

(51.) $\frac{4}{x^2+1} - \frac{2x+1}{(x^2+1)^2} + \frac{1}{(x^2+x+1)^2} - \frac{4}{x^2+x+1}$.

(52.) $\frac{1}{x^3} + \frac{1}{x^2} - \frac{2}{x} + \frac{x-1}{(x^2+1)^2} - \frac{2x-1}{x^2+1}$.

(53.) $\frac{x+y}{y} - \frac{x}{x+y} - \frac{x^3-x^2y}{x^2y-y^3} + \frac{(x-y)^2}{2xy}$.

(54.) $\frac{8}{5(y-2)} - \frac{13}{80(y+3)} - \frac{5}{4(y-1)^2} - \frac{23}{16(y-1)}$.

(55.) $\frac{x^{3n}}{x^n-1} - \frac{x^{2n}}{x^n+1} - \frac{1}{x^n-1} + \frac{1}{x^n+1}$.

(56.) $\left(\frac{1+x}{1-x} + \frac{4x}{1+x^2} + \frac{8x}{1-x^4} - \frac{1-x}{1+x}\right) \div \left(\frac{1+x^2}{1-x^2} + \frac{4x^2}{1-x^4} - \frac{1-x^2}{1+x^2}\right)$.

(57.) $\frac{1}{3(x-1)} - \frac{1}{x+1} - \frac{x-1}{2(x^2-x+1)} + \frac{7x-1}{6(x^2+x+1)}$.

(58.) $\frac{2}{(x+1)^2} + \frac{8}{x+1} - \frac{8x+1}{x^2+x+1} - \frac{1}{(x+1)^2}$.

(59.) $\frac{1}{2a(x-a)^2} + \frac{1}{4a^2(x-a)} + \frac{1}{12a^2(x+a)} - \frac{x+a}{3a^2(x^2-ax+a^2)}$.

(60.) $\frac{a^2-(b-c)^2}{(a+c)^2-b^2} + \frac{b^2-(c-a)^2}{(a+b)^2-c^2} + \frac{c^2-(a-b)^2}{(b+c)^2-a^2}$.

(61.) $\frac{bc(a+d)}{(a-b)(a-c)} - \frac{ac(b+d)}{(a-b)(b-c)} - \frac{ab(c+d)}{(a-c)(c-b)}$.

(62.) $\frac{a+b}{ab}(a^2+b^2-c^2) + \frac{b+c}{bc}(b^2+c^2-a^2) + \frac{a+c}{ac}(a^2+c^2-b^2)$.

(63.) $\frac{a}{b} - \frac{(a^2-b^2)x}{b^3} + \frac{a(a^2-b^2)x^2}{b^2(b+ax)} - \frac{a+bx}{b+ax}$.

(64.) $\frac{\frac{1}{a} + \frac{1}{ab^3}}{b - 1 + \frac{1}{b}} \div \frac{b^2+b+\frac{1}{4}}{2b+1}$.

(65.) $$\frac{a + \dfrac{b-a}{1+ba}}{1 - a\dfrac{b-a}{1+ba}} \times \frac{a^2}{b^2} \times \frac{a + b + \dfrac{b^2}{a}}{a + b + \dfrac{a^2}{b}}.$$

(66.) $$\frac{1}{a + \dfrac{1}{b + \dfrac{1}{c}}}; \qquad \frac{1}{x - 1 + \dfrac{1}{1 + \dfrac{x}{4-x}}}.$$

(67.) $$\frac{a+c}{(a-b)(x-a)} - \frac{b+c}{(a-b)(x-b)} - \frac{x+c}{(x-a)(x-b)}.$$

(68.) $$\frac{1}{(a-b)(a-c)(x+a)} + \frac{1}{(b-a)(b-c)(x+b)} + \frac{1}{(c-a)(c-b)(x+c)}.$$

(69.) $$\frac{a}{(a-b)(a-c)(x-a)} + \frac{b}{(b-a)(b-c)(x-b)} + \frac{c}{(c-a)(c-b)(x-c)}.$$

(70.) $$\frac{a^2}{(a-b)(a-c)(x-a)} + \frac{b^2}{(b-a)(b-c)(x-b)} + \frac{c^2}{(c-a)(c-b)(x-c)}.$$

(71.) $$\frac{a^2+a+1}{(a-b)(a-c)(x-a)} + \frac{b^2+b+1}{(b-a)(b-c)(x-b)} + \frac{c^2+c+1}{(c-a)(c-b)(x-c)}.$$

(72.) $$\frac{a^2+ma+n}{(a-b)(a-c)(x-a)} - \frac{b^2+mb+n}{(a-b)(b-c)(x-b)} + \frac{c^2+mc+n}{(a-b)(b-c)(x-c)}.$$

Reduce to their lowest terms:—

(73.) $$\frac{a^3-x^3}{a^3+x^3} \times \frac{a^2+x^2}{a^2-x^2} \times \frac{a+x}{a-x} \times \frac{a^2-ax+x^2}{a^2+ax+x^2}.$$

(74.) $$\frac{x^2+3x+2}{x^2+2x+1} \times \frac{x^2+5x+4}{x^2+7x+12}.$$

(75.) $\dfrac{1}{\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x-2}}+\dfrac{x}{3x^2-6x+2}.$

(76.) $\dfrac{1}{\dfrac{1}{x+3}+\dfrac{1}{x-5}+\dfrac{1}{x+7}}+\dfrac{x^2+10x+21}{3x^2+10x-29}.$

(77.) $\dfrac{x^2-4x+3}{x^2-2x-3}+\dfrac{x-1}{x+1}\times\dfrac{x^3+1}{x^2-x+1}.$

(78.) $\dfrac{x^2+(a-b)x-ab}{x^2+(a+b)x+ab}+\dfrac{x-b}{x+b}.$

(79.) $\dfrac{x^2+11x+30}{9x^3+53x^2-9x-18}+\dfrac{x+5}{9x^2-x-3}.$

(80.) $\dfrac{ac+bd+ad+bc}{af+2bx+2ax+bf}+\dfrac{c+d}{f+2x}.$

(81.) $\dfrac{a^3+ab^2-a^2b-b^3}{4a^4-2a^2b^2-4a^3b+2ab^3}+\dfrac{a^2+b^2}{2a(2a^2-b^2)}.$

(82.) $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}+\dfrac{x^{m-1}}{b(a+bx)}.$

(83.) $\dfrac{x^4+x^2y^2+y^4}{x^4+2x^3y+3x^2y^2+2xy^3+y^4}\times\dfrac{x^2+xy+y^2}{x^2-xy+y^2}.$

(84.) $\dfrac{x^4-2x^3y+2xy^3-y^4}{x^4-2x^3y+2x^2y^2-2xy^3+y^4}\times\dfrac{x^2+y^2}{x^2-y^2}.$

(85.) $\dfrac{6x^5+15x^4y-4x^3z^2-10x^2yz^2}{9x^3y-27x^2yz-6xyz^2+18yz^3}\div\dfrac{2x+5y}{x-3z}.$

(86.) $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}\div\dfrac{a+b+c}{a-b-c}.$

(87.) $\left(\dfrac{a^2}{bc}-\dfrac{2a}{d}+\dfrac{ac}{be}+\dfrac{bc}{d^2}-\dfrac{c^2}{de}\right)\div\left(\dfrac{a}{c}-\dfrac{b}{d}+\dfrac{c}{e}\right).$

(88.) $\left(\dfrac{3a^4}{x^4}-\dfrac{19a^3b}{10x^3y}+\dfrac{21a^2b^2}{5x^2y^2}-\dfrac{9ab^3}{10xy^3}+\dfrac{b^4}{y^4}\right)\div\left(\dfrac{a^2}{x^2}-\dfrac{ab}{2xy}+\dfrac{b^2}{y^2}\right).$

(89.) $\left(a^6+a^4+a^2+2+\dfrac{1}{a^2}+\dfrac{1}{a^4}+\dfrac{1}{a^6}\right)\div\left(a^3+\dfrac{1}{a^3}\right).$

SURDS.

---

*V. (1.) Multiply $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{1}{4}}$ by $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{1}{4}}$; $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{1}{4}}$ by $a^{\frac{1}{4}}b^{\frac{1}{3}}c^{\frac{1}{2}}$; and $a^{\frac{1}{2}}b^{\frac{1}{2}} - c^{\frac{1}{2}}x^{\frac{1}{2}}$ by $a^{\frac{1}{2}}b^{\frac{1}{2}} + c^{\frac{1}{2}}x^{\frac{1}{2}}$.

(2.) Multiply $3a^{\frac{1}{3}}b^{\frac{2}{3}} - 5a^{\frac{2}{3}}b^{\frac{4}{3}}$ by $3a^{\frac{1}{3}}b^{\frac{2}{3}} + 5a^{\frac{2}{3}}b^{\frac{4}{3}}$.

(3.) Multiply $a^{\frac{4}{5}} + a^{\frac{3}{5}}y^{\frac{1}{5}} + a^{\frac{2}{5}}y^{\frac{2}{5}} + a^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}}$ by $a^{\frac{1}{5}} - y^{\frac{1}{5}}$.

(4.) Multiply $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

(5.) Multiply $x^{\frac{3}{4}} - x^{\frac{1}{2}} + x^{\frac{1}{4}}$ by $x^{\frac{3}{4}} + x^{\frac{1}{2}} + x^{\frac{1}{4}}$.

(6.) Multiply $a^{\frac{3}{8}} + a^{\frac{1}{4}}b^{\frac{1}{8}} + a^{\frac{1}{8}}b^{\frac{1}{4}} + b^{\frac{3}{8}}$ by $a^{\frac{1}{8}} - b^{\frac{1}{8}}$.

(7.) Multiply $3x^{\frac{1}{2}} + 4x^{\frac{1}{3}} + 5x^{\frac{1}{6}}$ by $3x^{\frac{1}{2}} - 4x^{\frac{1}{3}} + 5x^{\frac{1}{6}}$.

(8.) Multiply $3x^{\frac{2}{5}} + 3x^{\frac{1}{5}}y^{\frac{1}{5}} + 3y^{\frac{2}{5}}$ by $x^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{1}{5}} + y^{\frac{2}{5}}$.

(9.) Multiply $a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} + b^{\frac{3}{4}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.

(10.) Multiply $x^{\frac{3}{4}} - m^{\frac{1}{4}}p^{\frac{1}{4}}x^{\frac{1}{2}} + m^{\frac{1}{2}}p^{\frac{1}{4}}x^{\frac{1}{4}} - m^{\frac{3}{4}}p^{\frac{1}{2}}$ by $x^{\frac{1}{4}} + p^{\frac{1}{4}}m^{\frac{1}{4}}$.

(11.) Divide $a^{\frac{3}{4}} - 3a^{\frac{1}{2}}b^{\frac{1}{4}} + 3a^{\frac{1}{4}}b^{\frac{1}{2}} - b^{\frac{3}{4}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.

(12.) Divide $1 - y^{\frac{3}{4}}$ by $1 - y^{\frac{1}{12}}$.

(13.) Divide $9ab^{\frac{4}{3}} - 25a^{\frac{2}{3}}b^{\frac{4}{3}}$ by $3a^{\frac{1}{2}}b^{\frac{2}{3}} + 5a^{\frac{1}{3}}b^{\frac{2}{3}}$.

(14.) Divide $x^{\frac{1}{3}} - 2 + x^{-\frac{1}{3}}$ by $x^{\frac{1}{6}} - x^{-\frac{1}{6}}$.

(15.) Divide $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ by $a^{\frac{1}{8}} - b^{\frac{1}{8}}$.

(16.) Divide $x^{6} - 3x^{2} + 3x^{-2} - x^{-6}$ by $x^{2} - x^{-2}$.

(17.) Divide $x^{\frac{1}{2}} + x^{-\frac{1}{2}} - x^{\frac{1}{4}} - x^{-\frac{1}{4}}$ by $x^{\frac{1}{8}} - x^{-\frac{1}{8}}$.

(18.) Divide $a^{\frac{1}{3}} + a^{\frac{1}{6}}x^{\frac{1}{6}} + x^{\frac{1}{3}}$ by $a^{\frac{1}{6}} + a^{\frac{1}{12}}x^{\frac{1}{12}} + x^{\frac{1}{6}}$.

(19.) Divide $x^{\frac{1}{3}} + \frac{x^{\frac{1}{6}}}{4} + \frac{1}{16}$ by $x^{\frac{1}{6}} - \frac{x^{\frac{1}{12}}}{2} + \frac{1}{4}$.

(20.) Divide $x^{\frac{1}{3}} + x^{-\frac{1}{3}} + 2(x^{\frac{1}{6}} - x^{-\frac{1}{6}}) - 1$ by $x^{\frac{1}{6}} - x^{-\frac{1}{6}} + 1$.

VI. (1.) Find the value of $\sqrt{32} + \sqrt{50} - \sqrt{18} + \sqrt{98}$.

Here $32 = 2 \times 16$, $50 = 2 \times 25$, $18 = 2 \times 9$, and $98 = 2 \times 49$.

$$\therefore \sqrt{32} + \sqrt{50} - \sqrt{18} + \sqrt{98}$$
$$= \sqrt{2 \times 16} + \sqrt{2 \times 25} - \sqrt{2 \times 9} + \sqrt{2 \times 49}$$
$$= 4\sqrt{2} + 5\sqrt{2} + 7\sqrt{2} - 3\sqrt{2}$$
$$= (4 + 5 + 7 - 3)\sqrt{2} = 13\sqrt{2}.$$

(2.) $3\sqrt{8} \times \sqrt{32} \times 3\sqrt{48} \times \sqrt{75}$.
$$= 3\sqrt{4 \times 2} \times \sqrt{2 \times 16} \times 3\sqrt{16 \times 3} \times \sqrt{25 \times 3}$$
$$= 3 \times 2 \times 16 \times 3 \times 5\sqrt{2 \times 2 \times 3 \times 3}$$
$$= 1440 \times 6 = 8640.$$

(3.) $3\sqrt{8} \times 2\sqrt{6} \times \sqrt{15} \times \sqrt{20}$.

(4.) $\sqrt[3]{81} \times \sqrt[3]{64} \times \sqrt[3]{375} \times \sqrt[3]{-24}$.

(5.) $4\sqrt{147} - 3\sqrt{75} - 6\sqrt{\frac{1}{3}}$.

(6.) $3\sqrt[3]{40} + \sqrt[3]{135} + \sqrt[3]{320}$.

(7.) $b\sqrt[3]{8a^6b} + 4a\sqrt[3]{a^3b^4} - \sqrt[3]{125a^6b^4}$.

(8.) $2\sqrt{128} - 2\sqrt{50} + 2\sqrt{72} - \sqrt{18}$.

(9.) $\sqrt{75} + \sqrt{147} + \sqrt{72} + \sqrt{128}$.

(10.) $\sqrt[3]{500} - \sqrt[3]{108} + \sqrt{243} + \sqrt{27} + \sqrt{75}$.

(11.) $\sqrt{128} + 2\sqrt{48} + \sqrt{338} + \sqrt{507}$.

(12.) $2\sqrt{8} - 7\sqrt{18} + 5\sqrt{72} - \sqrt{50}$.

(13.) $\dfrac{8\sqrt{3}}{4} - \dfrac{1}{4}\sqrt{48} + 4\sqrt{27} - \dfrac{2}{3}\sqrt{\dfrac{27}{16}}$.

(14.) $2\sqrt{8} - 7\sqrt{18} + 5\sqrt{72} + \sqrt{50}$.

(15.) $\sqrt{12} + 2\sqrt{27} + 3\sqrt{75} + 13\sqrt{3}$.

(16.) $2\sqrt[3]{4} \times 7\sqrt[3]{6} \times \frac{1}{2}\sqrt[3]{5} \times \sqrt[3]{100}$.

(17.) $\sqrt{\frac{2}{3}} \times \sqrt{\frac{5}{7}} \times \sqrt{\frac{3}{8}} \times \sqrt{\frac{20}{3}} \times \sqrt{\frac{245}{44}} \times \sqrt{\frac{14}{45}}$.

(18.) $\sqrt[3]{\frac{1}{2}} \times \sqrt[3]{\frac{1}{3}} \times \sqrt[3]{\frac{1}{4}} \times \sqrt[3]{\frac{3}{5}} \times \sqrt[3]{\frac{24}{25}} \times \sqrt[3]{\frac{250}{27}} \times \sqrt[3]{\frac{189}{70}}$.

(19.) $7\frac{3}{11}\sqrt{\dfrac{2}{7}} \times \dfrac{18}{15}\sqrt{\dfrac{14}{15}} \times \dfrac{77}{5\sqrt{15}}$.

(20.) $\frac{3}{7}\sqrt{\frac{5}{11}} + \sqrt{\frac{11}{5}} + \frac{1}{4}\sqrt{\frac{44}{245}}$.

(21.) $\left(\frac{a}{x}\right)^{\frac{1}{2}} \times \left(\frac{x}{b}\right)^{\frac{1}{2}} \times \left(\frac{b}{y}\right)^{\frac{1}{2}}.$

(22.) $\frac{3a}{4}\sqrt[3]{8a^3b} - \frac{3b}{2}\sqrt[3]{\frac{a^6b}{27b^3}} + a\sqrt[3]{a^3b}.$

(23.) $\sqrt{\frac{ay}{x}} \times \sqrt[3]{\frac{bx}{y^2}} \times \sqrt[6]{\frac{y^2}{b^2a^3}} \times \sqrt[6]{\frac{y}{x}}.$

(24.) $(\sqrt[3]{128} + \sqrt[3]{54} + \sqrt[3]{250}) \times (\sqrt[3]{54} - \sqrt[3]{128} + \sqrt[3]{250}).$

(25.) $9\sqrt{4a^2b + 4a^2x} - 4\sqrt{9a^2b + 9a^2x}.$

(26.) $2\sqrt{a^2b} + 3\sqrt{64bx^2}$, and $\sqrt{80a^4x} - \sqrt{20a^2x^3}.$

(27.) $\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{9}{32}}$, and $\sqrt{\frac{8}{27}} - \sqrt{\frac{1}{6}}.$

(28.) $\sqrt{100a^2b} + \sqrt{144a^2b} - \sqrt{289a^2b}.$

(29.) $\left(\sqrt[3]{a^{-\frac{1}{2}}} + \sqrt[3]{\sqrt{a^{\frac{1}{2}}b}}\right) \times \left(a^{-\frac{1}{6}} - a^{\frac{1}{12}}b^{\frac{1}{6}}\right).$

(30.) $(5-2\sqrt{-3})\,(5+2\sqrt{-3})$, and $(3-\sqrt{-5})\,(6+2\sqrt{-5}).$

(31.) $(2-2\sqrt{-3})\,(5-5\sqrt{-3})$, and $(5\sqrt{6}-3\sqrt{5})\,(5\sqrt{6}+3\sqrt{5}).$

(32.) $\sqrt{-a} \times \sqrt{-b}$, and $\sqrt[4]{-a} \times \sqrt[4]{-b}.$

(33.) $(x-1+\sqrt{2}) \times (x-1-\sqrt{2}) \times (x+2+\sqrt{3}) \times (x+2-\sqrt{3}).$

(34.) $\sqrt[m]{x+y} \times \sqrt[n]{x+y} \times \sqrt[m]{x-y} \times \sqrt[n]{x-y}.$

(35.) $(1-\sqrt{3}) \times (\sqrt{2}+\sqrt{3}) \times (1+\sqrt{3}) \times (\sqrt{2}-\sqrt{3}).$

(36.) $x\sqrt{12a^2x} + 2a\sqrt{27x^3} + 3\sqrt{48a^2x^5} - 20ax\sqrt{3x}.$

(37.) $b\sqrt[3]{8a^6b} + 4a\sqrt[3]{a^3b^4} - \sqrt[3]{125a^6b^4}.$

(38.) $\frac{\sqrt{ab^3}}{c} + \frac{1}{2c}\sqrt{a^3b-4a^2b^2+4ab^3}.$

(39.) $\frac{\frac{1}{2}\sqrt{\frac{1}{6}}}{\sqrt{\frac{2}{3}} + 3\sqrt{\frac{1}{6}}}$, and $\frac{\frac{1}{8}(\sqrt{5} \mp 1)}{\frac{1}{4\sqrt{2}}\sqrt{5 \pm \sqrt{5}}}.$

(40.) $\frac{\sqrt{\frac{1}{2}}(\sqrt{5 \mp \sqrt{5}})}{\frac{1}{2}(\sqrt{5} \pm 1)}$, and $\frac{\sqrt[3]{20}}{\sqrt[3]{4} - \sqrt[3]{2}} \times \frac{\sqrt[3]{16} - \sqrt[3]{4}}{\sqrt[3]{8}}.$

(41.) $(a+b)^{\frac{1}{m}} \times (a+b)^{\frac{1}{n}} \times (a-b)^{\frac{1}{m}} \times (a-b)^{\frac{1}{n}}.$

(42.) $\sqrt[3]{\frac{1}{x} - x^2 + 3(x - 1)}$.

(43.) $\dfrac{(x + y)\,x^{\frac{1}{3}}}{x - \sqrt[3]{x^2y} + \sqrt[3]{xy^2}}$.

(44.) $\sqrt[4]{\frac{x}{y}} \times \sqrt{\frac{y}{x}} \times \sqrt[6]{\frac{1}{x^3y^2}} \times \sqrt[15]{x^6y^5}$.

(45.) $\sqrt[m]{a^{3p}x^{2m-n}y^{5m+1}} \times \sqrt[m]{a^{m-3p}x^{n}y^{m-1}}$.

(46.) $\dfrac{ab}{b - c} \pm \sqrt{\dfrac{a^2b^2}{(b - c)^2} - \dfrac{a^2b}{b - c}}$.

(47.) $\left(x^m + \frac{1}{4}\sqrt[\frac{np}{2}]{b^p x^{2n}} - b^{\frac{1}{n}} x^{\frac{mp + 4}{2p}}\right)^{\frac{1}{2}}$.

(48.) $\dfrac{2x^2 - 2y^2 - \sqrt{2xy}}{\sqrt{2}\,(2x^2 - y^2)}$.

(49.) $\dfrac{\sqrt{18a^5b^3} + \sqrt{50a^3b^5}}{\sqrt{8ab^3} + \sqrt{2a^3b - 8a^2b^2 + 8ab^3}}$.

(50.) $\dfrac{1}{4(1 + \sqrt{x})} + \dfrac{1}{4(1 - \sqrt{x})} + \dfrac{1}{2(1 + x)}$.

(51.) $\dfrac{2x^2}{(1 - x^2)^{\frac{3}{2}}} + \dfrac{1}{(1 - x^2)^{\frac{1}{2}}}$.

(52.) $\dfrac{x^2}{a - \sqrt{a^2 - x^2}} - \dfrac{x^2}{a + \sqrt{a^2 + x^2}}$.

(53.) $\dfrac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}} + \dfrac{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$.

(54.) $\dfrac{1}{x + \sqrt{x^2 - 1}} + \dfrac{1}{x - \sqrt{x^2 - 1}}$.

(55.) $\left(\sqrt{\dfrac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\dfrac{a - \sqrt{a^2 - b}}{2}}\right)^2$.

(56.) $\left(\sqrt{\dfrac{x^5y}{3x^2 - 6\sqrt{3}\,.\,xy + 9y^2}} + \dfrac{y^2\sqrt{3xy}}{x - y\sqrt{3}}\right) + \sqrt{\dfrac{xy}{3}(x + y\sqrt{3})}$

(57.) $\left(a^{-\frac{2}{xy}} - 2b^{\frac{1}{2r}}a^{\frac{x-2r}{2xyr}} + a^{-\frac{y}{r}}\ b^{\frac{1}{r}}\right)^{\frac{1}{2}}$.

(58.) $\dfrac{8 - 5\sqrt{2}}{3 - 2\sqrt{2}}$.

$$\because (3 - 2\sqrt{2})\ (3 + 2\sqrt{2}) = 9 - 8,$$

$$\therefore \frac{8 - 5\sqrt{2}}{3 - 2\sqrt{2}} \times \frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}} = \frac{24 + \sqrt{2} - 20}{9 - 8} = 4 + \sqrt{2}.$$

(59.) $\dfrac{3 + \sqrt{5}}{4 - \sqrt{5}}$; $\dfrac{\sqrt{3} + 7\sqrt{5}}{\sqrt{3} + \sqrt{5}}$.

(60.) $\dfrac{6 + 10\sqrt{6}}{2\sqrt{3} + 3\sqrt{2}}$; $\dfrac{2(\sqrt{3} + 1)}{\sqrt{3} - \sqrt{2} + 1}$.

(61.) $\dfrac{2 + \sqrt{3}}{\sqrt{2} + \sqrt{2 + \sqrt{3}}} + \dfrac{2 - \sqrt{3}}{\sqrt{2} + \sqrt{2 - \sqrt{3}}}$.

(62.) $\dfrac{5 + 2\sqrt{-3}}{2 - \sqrt{-3}} + \dfrac{2}{2 + \sqrt{-3}} - \dfrac{4}{1 - \sqrt{-3}}$.

(63.) $\dfrac{1 + \sqrt{3}}{1 - \sqrt{3}} + \dfrac{2 + \sqrt{3}}{2 - \sqrt{3}} - \dfrac{2\sqrt{3} + 1}{2 + \sqrt{3}}$.

(64.) $\dfrac{a + \sqrt{-b}}{a - \sqrt{-b}} + \dfrac{a - \sqrt{-b}}{a + \sqrt{-b}}$.

(65.) $\dfrac{\sqrt[3]{x} + \sqrt{y}}{\sqrt[3]{x} - \sqrt{y}} - \dfrac{2x^{\frac{1}{3}}y^{\frac{1}{2}}}{x^{\frac{2}{3}} - y}$.

(66.) $\dfrac{a^2 + ab + b^2}{(a^3 - b^3)\ (x^{\frac{1}{2}} - a^{\frac{1}{2}})} - \dfrac{x^{\frac{1}{2}}}{(a - b)\ (x - a)}$.

(67.) $\sqrt{\dfrac{a + b - c}{(a + c - b)\ (b + c - a)}} + \sqrt{\dfrac{a + c - b}{(b + c - a)\ (a + b - c)}}$

$+ \sqrt{\dfrac{b + c - a}{(a + c - b)\ (a + b - c)}}$.

(68.) $\dfrac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{(a^{\frac{1}{2}} + b^{\frac{1}{2}})^3} + \dfrac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2} + \dfrac{a^{\frac{1}{2}} - b^{\frac{1}{2}}}{a^{\frac{1}{2}} + b^{\frac{1}{2}}}$.

(69.) $\dfrac{1}{x-1} + \dfrac{2}{2x+1-\sqrt{-3}} + \dfrac{2}{2x+1+\sqrt{-3}}$.

(70.) $\dfrac{\sqrt{2}+4\sqrt{3}}{\sqrt{3}+\sqrt{2}-\sqrt{5}}$; $\dfrac{1+\sqrt{2}}{1+\sqrt{2}-\sqrt{3}}$.

(71.) $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} - \dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

(72.) Find the root of $5 + 2\sqrt{6}$.

$$\text{Let } \sqrt{5+2\sqrt{6}} = \sqrt{x} + \sqrt{y}.$$

$$\text{Squaring both sides, } 5 + 2\sqrt{6} = x + y + 2\sqrt{xy};$$

$$\therefore x + y = 5, \text{ and } \sqrt{xy} = \sqrt{6};$$

$$\begin{aligned} \text{and } x^2 + 2xy + y^2 &= 25 \\ 4xy &= 24 \\ \hline \therefore x^2 - 2xy + y^2 &= 1; \end{aligned}$$

$$\text{and } x - y = 1, \text{ and } x + y = 5;$$

$$\therefore x = 3,\ y = 2, \text{ and } \sqrt{5+2\sqrt{6}} = \sqrt{3} + \sqrt{2}.$$

(73.) $7 + 2\sqrt{10}$; $8 + 2\sqrt{7}$.

(74.) $0{\cdot}03 + 0{\cdot}04\sqrt{-1}$; $37 - 20\sqrt{3}$.

(75.) $28 - 10\sqrt{3}$; $2\sqrt{-3} - 2$.

(76.) $21 - \sqrt{-400}$; $4\sqrt{-5} - 1$.

(77.) $2 + \sqrt{3}$; $-18\sqrt{-1}$.

(78.) $4 - \sqrt{7}$; $16 \pm 8\sqrt{3}$.

Find the fourth root of—

(79.) $97 + 28\sqrt{12}$; $\frac{17}{8} - 4\sqrt{2}$.

(80.) $14 + 8\sqrt{3}$; $-16a^4$.

Find the square root of—

(81.) $a^2 + 2x\sqrt{a^2 - x^2}$; $2 + 2(1-x)\sqrt{1 + 2x - x^2}$.

(82.) $1 + \sqrt{1 - m^2}$; $xy - 2x\sqrt{xy - x^2}$.

Find the cube root of—

(83.) $16 + 8\sqrt{5}$; $22 + 10\sqrt{7}$.

(84.) $11\sqrt{2} + 9\sqrt{3}$; $2\sqrt{7} + 3\sqrt{3}$.

(85.) $2\sqrt{-1} - 11$; $25 + 21\sqrt{3} + 17\sqrt{5} + 6\sqrt{15}$.

(86.) $x^{\frac{3}{2}} - 3x^{\frac{4}{3}} + 3x^{\frac{7}{6}} + 2x + 3x^{\frac{5}{4}} - 3x^{\frac{5}{6}} - 6x^{\frac{13}{12}} + 3x^{\frac{11}{12}} - x^{\frac{3}{4}}$.

(87.) Show that $2^{\frac{1}{3}} > 3^{\frac{1}{5}}$; $\sqrt{10} + \sqrt{7} < \sqrt{19} + \sqrt{3}$; $\sqrt[3]{2} + \sqrt[3]{5} < 3$; $\left(\frac{1}{2}\right)^{\frac{1}{2}} < \left(\frac{2}{3}\right)^{\frac{2}{3}}$.

(88.) Show that $2(1 + n^2 + n^4) > 3(n + n^3)$; $n^{\frac{3}{2}} - 1 > (n-1)^{\frac{3}{2}}$ unless $n = 1$.

(89.) Show that $a + \sqrt{a} > 1 + \sqrt{a^3}$, and that $a^{\frac{3}{2}} - 1 > a - 1$ if a be greater than unity.

(90.) Of the quantities $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[5]{5}$, which the greatest, and which is the least?

(91.) Show that $\sqrt[4]{-1} = \dfrac{1 + \sqrt{-1}}{\sqrt{2}}$, and that $\dfrac{a + b\sqrt{-1}}{c + d\sqrt{-1}} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}\sqrt{-1}$.

SIMPLE EQUATIONS.

VII. (1.) $\dfrac{x}{6} + \dfrac{11}{2} = \dfrac{x + 5}{2} + \dfrac{x - 8}{4} + \dfrac{1}{3}$.

(2.) $\dfrac{3x}{8} + 1 + \dfrac{1}{21} - \dfrac{2x}{7} = \dfrac{31}{28}$.

(3.) $\dfrac{25x + 5}{6} - \dfrac{8 + 2x}{5} = \dfrac{3x + 9}{4}$.

(4.) $\dfrac{x - 18}{4} + \dfrac{2x - 24}{11} + \dfrac{11x - 34}{22} = \dfrac{7}{44}$.

(5.) $\dfrac{2x + 12}{6} + \dfrac{2x - 15}{6} + \dfrac{x}{2} = 3$.

(6.) $\dfrac{3x}{4} + \dfrac{7x}{15} + \dfrac{11x}{6} - \dfrac{183}{5} = 0$.

(7.) $\frac{1}{x} + \frac{1}{2x} - \frac{1}{3x} = \frac{7}{24}$.

(8.) $10(x + \frac{1}{2}) - 6x\left(\frac{1}{x} - \frac{1}{3}\right) = 28$.

(9.) $x + \frac{11 - x}{3} = \frac{19 - x}{2} + \frac{7}{3} + \frac{2x - 14}{7}$.

(10.) $3x + 20 = 7 - \frac{1}{2}[3 - \frac{4}{5}(x - 1)]$.

(11.) $\frac{x}{2} + \frac{5x + 4}{3} = \frac{4x + 9}{3} + \frac{5}{12}$.

(12.) $\frac{x}{12} - \frac{8 - x}{8} - \frac{1}{4}(5 + x) + \frac{11}{4} = 0$.

(13.) $\frac{x}{8} - \frac{x - 1}{2\frac{1}{2}} = \frac{3x - 4}{15} + \frac{x}{12}$.

(14.) $\frac{1}{6}(x + 3) - \frac{1}{7}(11 - x) = \frac{2}{5}(x - 4) - \frac{1}{21}(x - 3)$.

(15.) $\frac{x + 1}{2} + \frac{x + 2}{3} = 16 - \frac{x - 6}{4}$.

(16.) $\frac{1}{7}(x - \frac{1}{2}) - \frac{1}{3}(\frac{2}{3} - x) = 1\frac{13}{36}$.

(17.) $\frac{x + 2}{3} + \frac{x + 3}{4} = 7 + \frac{x - 4}{6}$.

(18.) $\frac{2}{3}(x - 5) - \frac{3}{11}(x - 13\frac{1}{3}) = 15 - \frac{2}{5}(19 - \frac{1}{3}x)$.

(19.) $\frac{5x - 7}{3} - \frac{4x - 9}{5} + 2x = 13\frac{4}{5}$.

(20.) $3x - 4 - \frac{4}{5} \times \frac{7x - 9}{3} = \frac{4}{5} \times \frac{17 + x}{3}$.

(21.) $\frac{5x}{9} - \frac{2x - 1}{3} = \frac{4}{15} + \frac{5x - 3}{5}$.

(22.) $\frac{5x - 7}{3} - \frac{3x - 2}{7} = \frac{x - 5}{4} + \frac{83x - 67}{83}$.

(23.) $\frac{x}{2} - \frac{x - 2}{3} = \frac{1}{4}\left[x - \frac{2}{3}(2\frac{1}{2} - x)\right] - \frac{1}{3}(x - 5)$.

(24.) $\frac{5x - 4}{9} - \frac{2x - 13}{7} = \frac{x + 1}{3}$.

(25.) $3\frac{1}{3}\left[28 - \left(\frac{x}{8} + 24\right)\right] = 3\frac{1}{2}\left(2\frac{1}{3} + \frac{x}{4}\right)$.

(26.) $\frac{2}{3}(x - 5) - \frac{3}{11}(x - 14) = 5 - \frac{1}{5}(9 - x)$.

(27.) $\frac{x-1}{7} + \frac{23-x}{5} = 7\frac{81}{140} - \frac{x+4}{4}$.

(28.) $\frac{3x+4}{5} - \frac{7x-3}{2} - \frac{9x-16}{4} + \frac{183}{20} = 0$.

(29.) $8\frac{3}{4} + \frac{3x}{7} - \frac{5}{6} + 2x - \frac{12x}{5} + 13 + \frac{x}{4} = 22\frac{13}{15}$.

(30.) $\frac{3}{2}(x+2) - \frac{3}{7}[1\frac{1}{3} - (1\frac{1}{2} + x)] = 4\frac{20}{21}$.

(31.) $\frac{x+1}{2} + \frac{x+3}{3} = 14 + \frac{5-x}{4} + \frac{14x}{51}$.

(32.) $\frac{1}{14}(3x + \frac{2}{3}) - \frac{1}{7}(4x - 6\frac{2}{3}) = \frac{1}{2}(5x-6)$.

(33.) $\frac{x}{21} - \frac{x-7}{3} + \frac{3x-1}{5} - \frac{2x}{7} = 2\frac{11}{15}$.

(34.) $\frac{5x-1}{2} - \frac{7x-2}{10} = 29\frac{2}{5} - \frac{x}{2} + \frac{117}{10}$.

(35.) $\frac{4x-21}{7} + 7\frac{5}{6} + \frac{7x-28}{3} = x + 3\frac{3}{4} - \frac{9-7x}{8} + \frac{1}{12}$.

(36.) $\frac{2x}{3} - \frac{1-\frac{x}{2}}{4x} = \frac{x-1}{2} + \frac{x}{6} + \frac{7}{12}$.

(37.) $\frac{11x+12}{25} + \frac{19x+3}{7} - \frac{5x-25\frac{1}{3}}{4} + \frac{17x-17}{21} = 28\frac{1}{7}$.

(38.) $\frac{4x-17}{x-4} + \frac{10x-13}{2x-3} - \frac{8x-30}{2x-7} = \frac{5x-4}{x-1}$.

(39.) $\frac{x+3}{x+1} - \frac{x+4}{x+2} + \frac{x-6}{x-4} = \frac{x^2-2x-15}{x^2-9}$.

(40.) $\frac{7x+6}{28} - \frac{2x+4\frac{2}{7}}{23x-6} + \frac{x}{4} = \frac{11x}{21} - \frac{x^2-9}{42(x+3)}$.

(41.) $\frac{3x-1}{x-4} + \frac{2x-1}{x+4} - 5 = \frac{96}{x^2-16}$.

(42.) $\frac{3x}{2} + \frac{9(9x^2-1)}{(1-3x)(x+3)} + \frac{6x^2-3}{2(x+3)} = \frac{9x-57}{2}$.

(43.) $\frac{5x^2+x-3}{5x-4} - \frac{7x^2-8x-9}{7x-10} = \frac{x-3}{35x^2-78x+40}$.

(44.) $\frac{1}{1-x^2} + \frac{x+2}{4-x^2} - \frac{3+x}{9-x^2} = 0$.

(45.) $\dfrac{12x+2}{3x-2} + \dfrac{3x-2}{3x+2} = \dfrac{15x+11}{3x+2}$.

(46.) $\dfrac{1}{3x-1} + \dfrac{2(x+1)}{x^2-1} - \dfrac{16x^2+x+3}{4(3x^2-4x+1)} = \dfrac{1}{x}$.

(47.) $\dfrac{x+\dfrac{4}{3}}{x+\dfrac{1}{2}} - \dfrac{x+20}{x+12} - 1 = 0$.

(48.) $\dfrac{x+8}{x+12} - \dfrac{3x+14}{3x+8} + \dfrac{6x-24}{x^2-16} = 0$.

(49.) $\dfrac{25-\dfrac{x}{3}}{x+1} + \dfrac{16x+4\frac{1}{3}}{3x+2} = 5 + \dfrac{23}{x+1}$.

(50.) $\left(\dfrac{8x-3}{4x-1}\right)^2 - \dfrac{4x-5}{x-1} = 0$.

(51.) $\dfrac{x}{x-2} + \dfrac{x-9}{x-7} = \dfrac{x+1}{x-1} + \dfrac{x-8}{x-6}$.

(52.) $\dfrac{3}{1-3x} + \dfrac{5}{1-5x} + \dfrac{4}{2x-1} = 0$.

(53.) $\dfrac{x-3}{x^2-9} - \dfrac{12-2x}{x^2-36} = \dfrac{3x-27}{x^2-81}$.

(54.) $\dfrac{17-4x}{2} : \dfrac{15+2x}{3} - 2x :: 5 : 2$.

(55.) $\dfrac{2x+7}{9x+31} : \dfrac{16x+5}{2} :: 1 : 36x+10$.

(56.) $4x+3 : 6x-43 :: 2x+19 : 3x-19$.

(57.) $10+x : 4x-9 :: 2 : 1$.

(58.) $\cdot3x + 3\cdot15 - 1\cdot75x = \cdot125x$.

(59.) $\cdot6x + \cdot2 - \cdot7x + \cdot75x - \cdot875x + \cdot1 = 0$.

(60.) $\cdot375x + \cdot05 = \cdot225x + \cdot8$.

(61.) $\cdot05x - \cdot25 = \cdot075x - \cdot45$.

(62.) $2\cdot4x - \cdot072x + \cdot1 = \cdot8x + 9\cdot268$.

(63.) $\cdot6x + \cdot8 - 3\cdot5x + 1\cdot5 + 4 = \cdot25x$.

(64.) $\dfrac{\cdot 06x - \cdot 02}{4} = \dfrac{\cdot 036x - \cdot 005}{5} + \cdot 23.$

(65.) $\cdot 15x - \cdot 875x + 1\cdot 575 = \cdot 0625x.$

(66.) $\dfrac{5ab}{6} + \dfrac{4ac}{5} - \dfrac{2cx}{3} = \dfrac{3ac}{4} + 2ab - 6cx.$

(67.) $ax - \dfrac{a^2 - 3bx}{a} - ab^2 = bx + \dfrac{6bx - 5a^2}{2a} - \dfrac{bx + 4a}{4}.$

(68.) $(a + b)(b - x) + (a - b)(a + x) = c^2.$

(69.) $\dfrac{x - a}{b} - \dfrac{x - b}{a} = \dfrac{b}{a}.$

(70.) $\dfrac{c + x}{c - x} - \dfrac{c - x}{c + x} = \dfrac{5b^2}{4(c^2 - x^2)}.$

(71.) $\dfrac{a}{x - a} - \dfrac{a}{7(x - a)} = \dfrac{2a}{x + 7a}.$

(72.) $\dfrac{3a}{x} - \dfrac{2a}{x + a} = \dfrac{5a}{4(x + a)}.$

(73.) $\dfrac{x^2 + a^2}{4x^2 - a^2} - \dfrac{x}{2x + a} + \dfrac{1}{4} = 0.$

(74.) $\dfrac{2a}{x + a} + \dfrac{5a}{2x + 2a} - \dfrac{21}{8} = \dfrac{6(a - x)}{x^2 - a^2}.$

(75.) $\dfrac{7(a + x)^2}{5} - a(a + x) = \dfrac{6(a^2 - x^2)}{7} - \dfrac{16a^2}{35}.$

(76.) $\dfrac{20x + 11a}{25a} + \dfrac{5x + 20a}{9x - 16a} = \dfrac{4x}{5a} + \dfrac{61}{25}.$

(77.) $\dfrac{9x - 16a}{36a} = \dfrac{4x - 12a}{5x - 4a} + \dfrac{x - 4a}{4a}.$

(78.) $\dfrac{a + x}{a^2 + ax + x^2} + \dfrac{a - x}{a^2 - ax + x^2} = \dfrac{3a}{x(a^4 + a^2x^2 + x^4)}.$

(79.) $\dfrac{ad - bc}{d(c + dx)} + \dfrac{b}{d} = \dfrac{2a - bx}{c + dx}.$

(80.) $\dfrac{1}{(x - a)(x - c)} + \dfrac{2}{(a - c)(a - x)} = \dfrac{1}{(c - a)(c - x)}.$

(81.) $\dfrac{c}{a - b}\left(1 + \dfrac{1}{x}\right) - \dfrac{b}{a - c}\left(1 + \dfrac{1}{x}\right) = \dfrac{a + c}{(a - c)x} + 1.$

(82.) $\dfrac{1}{ab - ax} + \dfrac{1}{bc - bx} - \dfrac{1}{ac - ax} = 0.$

(83.) $\frac{x+c}{a+b} - \frac{2x}{(a+b)^2} = \frac{ac}{a^2-b^2} - \frac{b^2x}{a^3-ab^2+a^2b-b^3}$.

(84.) $\frac{a}{1+a^2} + \frac{x}{1+x^2} - \frac{(a+b)(1+ax)}{1+x^2} = 0$.

(85.) $\frac{x+a+b+c}{x^2+a^2+b^2+c^2} = \frac{1}{x+a+b+c}$.

(86.) $x+a : x-b :: (2x+a)^2 : (2x-b)^2$.

(87.) $4x+a : 4x-b :: (2x+a)^{\frac{1}{2}} : (2x-b)^{\frac{1}{3}}$

(88.) $a : b :: (x+2a+b)^2 : (x+2b+a)^2$.

(89.) $2(x+12)^{\frac{1}{2}} = 1$.

(90.) $(10x+35)^{\frac{1}{3}} = 5$.

(91.) $(9x-4)^{\frac{1}{3}} = 2$.

(92.) $\sqrt{x+16} = 2+\sqrt{x}$.

(93.) $\sqrt{4x+21} = 1+2\sqrt{x}$.

(94.) $\sqrt{16+x} = 2\sqrt{6+x}$.

(95.) $8\sqrt[3]{7x-6} = 16$.

(96.) $\sqrt{8+x}+\sqrt{x} = 2\sqrt{1+x}$.

(97.) $\sqrt{x+9} = 1+\sqrt{x}$.

(98.) $\frac{\sqrt{x}+28}{\sqrt{x}+38} = \frac{\sqrt{x}+4}{\sqrt{x}+6}$.

(99.) $\frac{5x-9}{\sqrt{5x}-3} = \frac{\sqrt{5x}+3}{2} + \frac{\sqrt{5x}+3}{\sqrt{5x}-3}$.

(100.) $x-\sqrt{a} = \sqrt{ax+x^2}$.

(101.) $a+x+\sqrt{2ax+x^2} = b$.

(102.) $\sqrt{4a+x}+\sqrt{x} = 2\sqrt{a+x}$.

(103.) $\sqrt{1+x+x^2}+\sqrt{1-x+x^2} = a$.

(104.) $bx\sqrt{a+x}+ab\sqrt{a+x} = ax^{\frac{3}{2}}$.

(105.) $x\sqrt{a^2+x^2}+x^2 = (n^2-1)a^2$.

(106.) $x-\sqrt{x^2-x} = (a-1)\sqrt{x}$.

(107.) $ax+\sqrt{a^2x^2+b^2} = \sqrt{b^2+\sqrt{a^2x^2(4b^2+x^2)}}$.

(108.) $a^2\sqrt{1-x}-\sqrt{a^2-x} = \sqrt{(a^2-1)x}$.

(109.) $\sqrt{b}(\sqrt{x^2+3a^2}-\sqrt{x^2-3a^2}) = 2x\sqrt{a}$.

(110.) $\sqrt{x+a}-\sqrt{x-a} = \sqrt{a}$.

(111.) $\sqrt{x}-\sqrt{n-\sqrt{nx+x^2}} = \sqrt{n}$.

(112.) $\sqrt{2x - 45} = 3\sqrt{15} - \sqrt{2x}.$

(113.) $\dfrac{1 - ax}{1 + ax}\sqrt{\dfrac{1 + bx}{1 - bx}} = 1.$

(114.) $(\sqrt{1 - x} - 1)(\sqrt{1 + x} + 1) = \sqrt{1 - x^2}.$

(115.) $\sqrt{x + 4a + 4b} + \sqrt{x} = 2\sqrt{b + x}.$

(116.) $\left(\dfrac{a + x}{a - x}\right)^2 - 1 = \dfrac{cx}{ab}.$

(117.) $(a + b)x = (a - b)\sqrt{1 + x^2}.$

(118.) $a + x = \sqrt{a^2 + x\sqrt{4c^2 + x^2}}.$

(119.) $(m + n\sqrt{x})(q + p\sqrt{x}) = (n + m\sqrt{x})(p + q\sqrt{x}).$

(120.) $x + \sqrt{x^2 - 2ax + b^2} = a + b.$

(121.) $(a - 1)(1 + x + x^2)^2 = (a + 1)(1 + x^2 + x^4).$

(122.) $x^2 + \sqrt{4x^2 + x + \sqrt{9x^2 + 12x}} = (1 + x)^2.$

(123.) $a + x - \sqrt{ax - x^2} = \sqrt{2a^2 - ax - x^2} - \sqrt{2ax + x^2}.$

(124.) $\sqrt{x^2 + 1 + x\sqrt{x - a - 4}} = 1 + x.$

(125.) $4x\left(x + \dfrac{a}{4}\right)^{\frac{1}{2}} = a(\sqrt{a} + \sqrt{a + 4x}).$

(126.) $\sqrt{x + \sqrt{x}} - \dfrac{a}{b}\sqrt{\dfrac{x}{x + \sqrt{x}}} = \sqrt{x - \sqrt{x}}.$

(127.) $x + a : x - a :: \sqrt{x^2 - ax + 2a^2} : \sqrt{x^2 - 5ax + 14a^2}.$

(128.) $\dfrac{ax - b^2}{\sqrt{ax} + b} = \dfrac{\sqrt{ax} - b}{n} - c.$

(129.) $\dfrac{\sqrt{1 - x^3}}{\sqrt{1 - x}} + \dfrac{\sqrt{1 + x^3}}{\sqrt{1 + x}} = mx.$

(130.) $(1 + x)^{\frac{1}{3}} + (1 - x)^{\frac{1}{3}} = 2^{\frac{1}{3}}.$

(131.) $\sqrt[3]{a + x} + \sqrt[3]{a - x} = b.$

(132.) $\sqrt{x^2 + 2ax} + \sqrt{x^2 - 2ax} = \dfrac{ax}{\sqrt{x^2 + 2ax}}.$

(133.) $\dfrac{1 + x^3}{(1 + x)^2} + \dfrac{1 - x^3}{(1 - x)^2} = a.$

(134.) $\dfrac{x + \sqrt{1 + x^2}}{2a\sqrt{1 + x^2}} = \dfrac{1}{a + b}$.

(135.) $\dfrac{\sqrt{a + \sqrt{a^2 - x^2}}}{\sqrt{a + x}} + \dfrac{\sqrt{a - \sqrt{a^2 - x^2}}}{\sqrt{a + x}} = n\left(a + \sqrt{a^2 - x^2}\right)^{-\frac{1}{2}}$.

(136.) $\dfrac{x}{\sqrt{1 - x} + 1} + \dfrac{x}{\sqrt{1 + x} - 1} = 1$.

(137.) $\dfrac{a(1 - x)}{\sqrt{1 + a^2}} + 1 = \dfrac{a\sqrt{1 + x^2}}{\sqrt{1 + a^2}}$.

(138.) $\sqrt{\dfrac{3a}{4} - x} + \sqrt{3ax - x} = \dfrac{3a}{2}\sqrt{1 - 4x}$.

(139.) $\dfrac{1 + x}{1 + x + \sqrt{1 + x^2}} + \dfrac{1 - x}{1 - x + \sqrt{1 + x^2}} = a$.

(140.) $\dfrac{1 + x + \sqrt{2x + x^2}}{1 - x + \sqrt{2x + x^2}} = 1 - ax$.

(141.) $\left(\dfrac{1 + x}{1 - x}\right)^{\frac{1}{4}} + \sqrt{\dfrac{1 - a}{1 + a}}\sqrt[4]{\dfrac{1 - x}{1 + x}} = 2 \times \sqrt[4]{\dfrac{1 - a^2}{(1 + a)^2}}$.

(142.) $\dfrac{1 + x - \sqrt{2x + x^2}}{1 + x + \sqrt{2x + x^2}} = a \times \dfrac{\sqrt{2 + x} + \sqrt{x}}{\sqrt{2 + x} - \sqrt{x}}$.

(143.) $\dfrac{a\sqrt{x} + \sqrt{1 + a^2x}}{2a\sqrt{1 + a^2x}} = \dfrac{1}{a + x}$.

(144.) $\dfrac{a - \sqrt{2ax - x^2}}{a + \sqrt{2ax - x^2}} = b$.

(145.) $\dfrac{1}{x^{\frac{1}{2}}} + \dfrac{1}{a^{\frac{1}{2}}} - \left(\dfrac{1}{a} + \sqrt{\dfrac{4}{ax} + \dfrac{9}{x^2}}\right)^{\frac{1}{2}} = 0$.

(146.) $\dfrac{a + x}{a + \sqrt{a^2 + ax}} + \dfrac{a - x}{a + \sqrt{a^2 - ax}} = 1$.

(147.) $\dfrac{\sqrt{1 + x} - 1}{\sqrt{1 - x} + 1} + \dfrac{\sqrt{1 - x} + 1}{\sqrt{1 + x} - 1} = a$.

(148.) $\sqrt[4]{x^4 - 1} + x\sqrt{x^4 - 1} = x^3$.

(149.) $\dfrac{\sqrt{a}-\sqrt{a-\sqrt{a^2-ax}}}{\sqrt{a}+\sqrt{a-\sqrt{a^2-ax}}}=b.$

(150.) $\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2-x^2}+a}=b.$

(151.) $\dfrac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}}=\dfrac{\sqrt{x^2-1}-1}{x^2-2}.$

(152.) $\dfrac{\sqrt{a+bx^n}+\sqrt{a-bx^n}}{\sqrt{a+bx^n}-\sqrt{a-bx^n}}=c.$

(153.) $\sqrt{a+x}+\sqrt{a-x}=\sqrt[4]{a^2+x^2}+\sqrt[4]{a^2-x^2}.$

(154.) $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9.$

(155.) $\sqrt{(1+x)^2-ax}+\sqrt{(1-x)^2+ax}=x.$

(156.) $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}}=1.$

(157.) $(1+x)\sqrt{1+a}+(1-x)\sqrt{1-a}=2\sqrt{1+x^2}.$

(158.) $(x-a)\sqrt{x}-(x+a)\sqrt{b}=b(\sqrt{x}-\sqrt{b}).$

(159.) $2x^2+1+x\sqrt{4x^2+3}=a(2x^2+3+x\sqrt{4x^2+3}).$

(160.) $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b^2.$

(161.) $\dfrac{243+324\sqrt{3x}}{(4\sqrt{x}-\sqrt{3})^2}=16x-3.$

(162.) $\dfrac{(a-x)+\sqrt{2ax-x^2}}{a-x}=b.$

(163.) $\sqrt[4]{x^4-1}+x\sqrt{x^4-1}=2x^{-1}.$

(164.) $ax+1=\dfrac{2ax\sqrt{x+a^2}}{a+\sqrt{x+a^2}}.$

(165.) $\dfrac{2a\sqrt{1+x^2}}{a+b}=1-x+\sqrt{1+x^2}.$

(166.) $\sqrt{(1+a)^2 + (1-a)x} + \sqrt{(1-a)^2 + (1+a)x} = 2a.$

(167.) $\dfrac{a^2(a+x) - a - x\sqrt{2a^2-1}}{a^2(a+x) - a + x\sqrt{2a^2-1}} = a^2 + x^2 - a^2(a+x)^2.$

QUADRATIC EQUATIONS.

VIII.

(1.) $12x^2 - 44 = 6x^2 + 10.$

(2.) $4x^2 - 4 = 28 + 2x^2.$

(3.) $(x+2)^2 - 5 = 4x.$

(4.) $3x^2 + 63 = 10x^2.$

(5.) $\dfrac{3}{x+1} - 8 = \dfrac{3}{x-1}.$

(6.) $\dfrac{2x^2+10}{15} + \dfrac{50+x^2}{25} = 7.$

(7.) $x^2 - 12x + 20 = 0.$

(8.) $x^2 - 10x + 16 = 0.$

(9.) $x^2 + 32x = 320.$

(10.) $2x^2 - 4x = 6.$

(11.) $x^2 + 13x + 12 = 0.$

(12.) $x^2 - 13x = 68.$

(13.) $x^2 + 25x + 100 = 0.$

(14.) $x^2 + 19x = 20.$

(15.) $3x^2 - x = 102.$

(16.) $20x^2 + 9 = 36x.$

(17.) $\dfrac{x^2}{3} + \dfrac{3x}{2} = 21.$

(18.) $\dfrac{2x^2}{3} - \dfrac{x}{2} = 4\tfrac{1}{2}.$

(19.) $6x^6 - 12x^3 = 288.$

(20.) $5x^4 - 11x^2 = 306.$

(21.) $x^6 - 4x^3 = 32.$

(22.) $3x^6 + 42x^3 = 3321.$

(23.) $x^2(x^2 - 18) = 4(x^4 - 12).$

(24.) $x^4 - 5x^2 + 6 = 0.$

(25.) $\dfrac{4}{x-3} - \dfrac{3}{x+5} = \dfrac{11}{3}.$

(26.) $\dfrac{2x}{x-4} + \dfrac{2x-5}{x-3} = 8\tfrac{1}{3}.$

(27.) $\dfrac{3x}{x+1} + \dfrac{2x-5}{3x-1} = \dfrac{217}{69}.$

(28.) $\dfrac{2x+3}{10-x} = \dfrac{2x}{25-3x} - 6\tfrac{1}{2}.$

(29.) $\dfrac{7}{x^2+4x} + \dfrac{21}{3x^2-8x} = \dfrac{22}{x}.$

(30.) $\dfrac{4x-35}{5} + \dfrac{36-5x}{5x} = 0.$

(31.) $\dfrac{x+3}{2} + \dfrac{16-2x}{2x-5} = 5\tfrac{1}{8}.$

(32.) $\dfrac{3x-7}{x} + \dfrac{4x-10}{x+5} = 3\tfrac{1}{2}.$

(33.) $\dfrac{x+2}{x-1} - \dfrac{7-2x}{2x} = \dfrac{7}{8}.$

(34.) $\dfrac{x^2}{4} = \dfrac{x^4-12}{x^2-18} + 126.$

(35.) $\dfrac{4x}{5-x} - \dfrac{20-4x}{x} = 15.$ (36.) $\dfrac{x^2-4}{x+2} = \dfrac{x^3-8}{2(x^2-10)}.$

(37.) $\dfrac{x+3}{x-1} + \dfrac{x-4}{2x} = 2\frac{1}{3}.$ (38.) $\dfrac{4(x^2-1)}{x+1} - \dfrac{x-1}{2x} = \dfrac{15}{4}.$

(39.) $\dfrac{x-1}{x^2-1} + \dfrac{x^2-1}{x-1} = \dfrac{13}{6}.$ (40.) $\dfrac{x^2-4}{x-2} - 1 = \dfrac{6(x-2)}{x^2-4}.$

(41.) $\dfrac{7+x}{7-x} + \dfrac{7-x}{7+x} = \dfrac{29}{10}.$ (42.) $\dfrac{3x+5}{3x-5} - \dfrac{135}{176} = \dfrac{3x-5}{3x+5}.$

(43.) $\dfrac{x^2}{(x^2-4)^2} = \dfrac{351}{25x^2} - \dfrac{6}{x^2-4}.$ (44.) $\dfrac{x+7}{x-7} - \dfrac{7x}{x^2-73} = \dfrac{x-7}{x+7}.$

(45.) $x^2 + \dfrac{1}{x^2} + x + \dfrac{1}{x} = 4.$ (46.) $x^2 + \dfrac{4}{x^2} + 6\left(x + \dfrac{2}{x}\right) = 23.$

(47.) $(3x^2+1)^2 - 9(3x^2+1) = 680.$

(48.) $(x + 8x^{-1})^2 + x = 42 - 8x^{-1}.$

(49.) $(x^2+5)^2 - 4x^2 = 160.$ (50.) $(x-5)^3 - 3(x-5)^{\frac{3}{2}} = 40.$

(51.) $x^2 - (a+b)x + ab = 0.$ (52.) $2b^2x^2 + ab(x-1) = 2b^2x.$

(53.) $adx - acx^2 = bcx - bd.$

(54.) $bc(x-a)^2 - bc = (b^2-c^2)(x-a).$

(55.) $1 + b^2x^2 = \dfrac{b}{a^2} \times (2a^2x + b).$

(56.) $x^2 - (a+b-c)x = (a+b)c.$

(57.) $a(x^2-ab) = b(x^2-2ax).$ (58.) $a^2bx + b^2x^2 = 2a^4.$

(59.) $9a^4b^4x^2 - 6a^3b^2x - b^2 = 0.$ (60.) $8x(a-x) = a(3a-2x).$

(61.) $a(1+2x) - a(1-2x) = a^2 - 4x^2.$

(62.) $mqx^2 - mnx + pqx = np.$ (63.) $8x^2 + 6(a-2b)x = 9ab.$

(64.) $6x^2 + 9(a-8b)x = 108ab.$ (65.) $9x^2 - 6bx = a^2 - b^2.$

(66.) $25x^2 - 10bx = a^2 - b^2.$ (67.) $4x^2 - 4bx - a^2 + b^2 = 0.$

(68.) $6a^2x^2 - 5abx = b^2.$

(69.) $4x^2 - 4(3a+2b)x + 24ab = 0.$

(70.) $mnx^2 - (n^2-m^2)x - mn = 0.$

(71.) $4x^2 - 4mx = n^2 - m^2.$ (72.) $4x^2 - 12ax + 9a^2 - 4b^2 = 0.$

(73.) $px^2 + qx + 1 = 0.$ (74.) $x^2 \pm px + q = 0.$

(75.) $(a-b)x^2-(a+b)x+2b=0.$

(76.) $anx^2+ab=na^2x+bx.$

(77.) $x^2-\frac{cx}{a+b}-\frac{2c^2}{(a+b)^2}=0.$ (78.) $x^2-ax+\frac{a^2-b^2}{4}=0.$

(79.) $\frac{a^2}{b+x}+\frac{a^2}{b-x}=\frac{c}{x}.$ (80.) $\frac{1}{a}+\frac{1}{a+x}+\frac{1}{a+2x}=0.$

(81.) $bx^2-\frac{ba}{b+c}=dx-cx^2.$ (82.) $\frac{1}{a}+\frac{1}{a-x}-\frac{5}{a-2x}=0.$

(83.) $\left(\frac{x}{x-1}\right)^2+\left(\frac{x}{x+1}\right)^2=n(n-1).$

(84.) $\frac{(x-a)(x-c)}{a+b-c}=b.$ (85.) $\frac{(x+a)^3}{(x-a)^2}-\frac{(x-a)^3}{(x+a)^2}=10a.$

(86.) $\frac{x^2+1}{a^2+3ab+b^2}=\frac{2x}{a^2+ab+b^2}.$

(87.) $\frac{6(a^2-b^2)}{x^2-1}=\frac{12(a^2+b^2)x}{x^4-1}+\frac{5ab}{x^2+1}.$

(88.) $(a-b)x^2+2b=(a+b)x.$

(89.) $\sqrt{a}(nx+b)=\sqrt{x}(na+b).$

(90.) $(x-c)\sqrt{ab}=(a-b)\sqrt{cx}.$

(91.) $abx^2+2\sqrt{ab}(a+b)x+(a-b)^2=0.$

(92.) $x-15=\sqrt{x}+5.$ (93.) $\sqrt{x}-2=x-8.$

(94.) $\sqrt{28x^2+39x+5}=30.$ (95.) $x+2=\sqrt{4+x\sqrt{8-x}}.$

(96.) $\sqrt{x^4+3x^2}+x^4=6-3x^2.$

(97.) $2x^2-11x+14\sqrt{11x-2x^2+2}=42.$

(98.) $x^2+11+\sqrt{x^2+11}=42.$

(99.) $9x+4=15x^2+2x\sqrt{9x+4}.$

(100.) $3x=8\sqrt{x+1}-7.$

(101.) $2x^2-2x+6\sqrt{2x^2-3x+2}=x+14.$

(102.) $(a^2-x^2)+a\sqrt{a^2-x^2}=12a^2.$

(103.) $ax+a\sqrt{x^2-ax+b^2}=x^2+ab.$

(104.) $x-15=\frac{x-9}{\sqrt{x}+3}.$ (105.) $\frac{x-4}{\sqrt{x}+2}=x-8.$

(106.) $(a-3n)x+n+2\sqrt{n^2x+nax^2}=0.$

(107.) $x^2-3x=\sqrt{x^2-3x+5}+1.$

(108.) $3x^2+2\sqrt{3x^2+3x}=48-3x.$

(109.) $x-1=2\sqrt{x-x^2}.$

(110.) $9x-4x^2+\sqrt{4x^2-9x+11}=5.$

(111.) $2x^2+\sqrt{x^2-9}=x^2+21.$ (112.) $x\sqrt{6-x^2}=x^2-6.$

(113.) $25x^2-31=25\sqrt{1-x^2}.$ (114.) $x\sqrt{6+x^2}=1+x^2.$

(115.) $\dfrac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}=(x-2)^2.$ (116.) $ab+b\sqrt{a^2-x^2}=x^2.$

(117.) $\dfrac{x^2}{a-\sqrt{a^2-x^2}}-\dfrac{x^2}{a+\sqrt{a^2-x^2}}=a.$

(118.) $\dfrac{x+\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}}=\dfrac{2a^2}{\sqrt{a^4-x^4}}.$

(119.) $x+a+2\sqrt{ax}=bx.$

(120.) $\sqrt{1+x+x^2}+\sqrt{1-x+x^2}=mx.$

(121.) $nx=(\sqrt{1+x}-1)(\sqrt{1-x}+1).$

(122.) $(x+a)^{\frac{2}{3}}=a^{\frac{1}{3}}(3x-a).$ (123.) $\dfrac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\dfrac{5}{11}.$

(124.) $\dfrac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}=\dfrac{1}{x^2}.$ (125.) $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}=\dfrac{1+\sqrt{1+x}}{1-\sqrt{1-x}}.$

(126.) $\dfrac{ax+1+\sqrt{a^2x^2-1}}{ax+1-\sqrt{a^2x^2-1}}=\dfrac{b}{x}.$

(127.) $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=b^2.$

(128.) $\dfrac{\sqrt{a+bx^n}+\sqrt{a-bx^n}}{\sqrt{a+bx^n}-\sqrt{a-bx^n}}=m.$

(129.) $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\dfrac{b}{x}.$

(130.) $\dfrac{x}{x+4} + \dfrac{4}{\sqrt{x+4}} = \dfrac{21}{x}.$

(131.) $\dfrac{x-18}{x^{\frac{1}{2}} - 18^{\frac{1}{2}}} + \dfrac{(x-18)^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \dfrac{x^{\frac{1}{2}}}{(x-18)^{\frac{1}{2}}} = x^{\frac{1}{2}}.$

(132.) $\dfrac{250}{7x\sqrt{x^2-9}} + \dfrac{\sqrt{x^2-9}}{7x} - \dfrac{19}{2x} = 0.$

(133.) $\dfrac{x+\sqrt{x}}{x^2-x} = \dfrac{x-\sqrt{x}}{4}.$

(134.) $x + 4 + \sqrt{\dfrac{x+4}{x-4}} = \dfrac{12}{x-4}.$

(135.) $x^3 - 3x^{\frac{3}{2}} = 40.$

(136.) $\dfrac{5(x-4)}{\sqrt{x}+2} = (x^{\frac{3}{2}} - 8).$

(137.) $x^3 - 1 = 2x^2 - 2x.$

(138.) $x^3 + 1 = 2x^2.$

(139.) $x^3 - 6x^2 + 10x = 5.$

(140.) $x^2 + \dfrac{64}{x^2} - \dfrac{5x-30}{2} = \dfrac{25x^2}{16}.$

(141.) $8x^4 + 8x = 16x^3 + 2.$

(142.) $4x^4 - 4x^3 + 4 = 2x - 5x^2.$

(143.) $x^4 - 2x^3 + x - 1 = 0.$

(144.) $x^4 - 2x^3 + x - 132 = 0.$

(145.) $x^{12} - 1 = 0.$

(146.) $2x^3 - x^2 = 1.$

(147.) $x^3 + px^2 + px + 1 = 0.$

(148.) $x^3 - px - p + 1 = 0.$

(149.) $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0.$

(150.) $x^4 + 5x^3 + 2x^2 + 5x + 1 = 0.$

(151.) $x^4 - \dfrac{5x^3}{2} + 2x^2 - \dfrac{5x}{2} + 1 = 0.$

(152.) $x^4 - \dfrac{5x^3}{2} + 3x^2 - \dfrac{5x}{2} + 1 = 0.$

(153.) $x^4 + x^3 + x + 1 = 4x^2.$

(154.) $a + x + \sqrt{2ax + x^2} = \dfrac{15x^2 + 6ax}{4\sqrt{2ax + x^2}}.$

(155.) $\dfrac{x}{\sqrt{x} + \sqrt{a-x}} + \dfrac{x}{\sqrt{x} - \sqrt{a-x}} = \dfrac{b}{\sqrt{x}}.$

(156.) $\sqrt{1+a}\sqrt{1-x} - \sqrt{1-a}\sqrt{1+x} = 2a.$

(157.) $\sqrt{x^4-1}+\sqrt{x^2-1}=x^3.$

(158.) $\sqrt[3]{(a+x)^2}+\sqrt[3]{(a-x)^2}-3\sqrt[3]{a^2-x^2}=0.$

(159.) $2x^{\frac{3}{2}}(a^3+x^3)^{\frac{1}{2}}-2x^2(x+2a)=a^2(x-a).$

(160.) $\dfrac{1}{x}+\dfrac{1}{a}=\sqrt{\dfrac{1}{a^2}+\sqrt{\dfrac{1}{a^2x^2}+\dfrac{5}{x^4}}}.$

(161.) $\dfrac{x^2}{8a}-\dfrac{2a}{3}=\sqrt{\dfrac{x^3}{3a}+\dfrac{x^2}{4}-\dfrac{4\sqrt{ax}}{3}-\dfrac{3x}{32}}.$

(162.) $\sqrt{3+\sqrt{x}}+\sqrt{4-\sqrt{x}}=\sqrt{7+2\sqrt{x}}.$

(163.) $(a+x)\sqrt{a^2+x^2}=6(a-x)^2.$

(164.) $\sqrt{x^2+1}-\sqrt{x^2-1}=x(x^4-1)^{-\frac{1}{2}}.$

(165.) $\dfrac{16-4\sqrt{x}}{8-3\sqrt{x}}=\dfrac{88+33\sqrt{x}}{4+\sqrt{x}}+\dfrac{x^2-5x+11}{(x-3\sqrt{x})(4+\sqrt{x})}.$

(166.) $\dfrac{123+41\sqrt{x}}{(5\sqrt{x}-x)}+\dfrac{2x^3}{(5\sqrt{x}-x)(3-\sqrt{x})}=\dfrac{20\sqrt{x}+4x}{3-\sqrt{x}}.$

(167.) $\dfrac{54-9\sqrt{x}}{x+2\sqrt{x}}-\dfrac{7x^2-3x+4}{(x+2\sqrt{x})\times(6+\sqrt{x})}=\dfrac{23x-46\sqrt{x}}{6+\sqrt{x}}.$

(168.) $\sqrt{x+\sqrt{2x-1}}-\sqrt{x-\sqrt{2x-1}}=\dfrac{3\sqrt{10x}}{5\sqrt{x+\sqrt{2x-1}}}.$

(169.) $\dfrac{x}{a+x}+\dfrac{a}{\sqrt{a+x}}=\dfrac{6a^2}{x}.$

(170.) $\sqrt{12-\dfrac{12}{x^2}}-x^2+\sqrt{x^2-\dfrac{12}{x^2}}=0.$

(171.) $\dfrac{2x+\sqrt{x}}{2x-\sqrt{x}}=\dfrac{52}{15}-3\,\dfrac{2x-\sqrt{x}}{2x+\sqrt{x}}.$

(172.) $\sqrt[2pq]{x^{p+q}}-\frac{1}{2}\,\dfrac{a^2-b^2}{a^2+b^2}\left(x^{\frac{1}{p}}+x^{\frac{1}{q}}\right)=0.$

(173.) $\dfrac{x^{(m-n)^2}+x^{-4mn}}{x^{(m-n)^2}-x^{-4mn}}=a^{\frac{r}{s}}.$

(174.) $a^2b^2x^{\frac{1}{n}} - 4 \times (ab)^{\frac{3}{2}} \times x^{\frac{m+n}{2mn}} = (a-b)^2x^{\frac{1}{m}}$.

(175.) $(a^{4m} + 1)(x^{\frac{1}{2}} - 1)^2 = 2(x + 1)$.

(176.) $x^{3m} - a = x^m(a + 1)$.

(177.) $(1 + x)^{\frac{2}{m}} - (1 - x)^{\frac{2}{m}} = (1 - x^2)^{\frac{1}{m}}$.

(178.) $a[(1 - x)\sqrt{x^2 + x^3} - (1 + x)\sqrt{x^2 - x^3}] = \sqrt{2 + 2\sqrt{1 - x^2}}$.

(179.) $\sqrt[3]{\sqrt{a} + \sqrt{x}} + \sqrt[3]{\sqrt{a} - \sqrt{x}} = \sqrt[6]{a}$.

(180.) $\dfrac{x}{2} - 2 = \dfrac{x^2}{2(1 + \sqrt{1 + x})^2}$.

(181.) $4x^2 + 12x\sqrt{1 + x} = 27(1 + x)$.

(182.) $(a + x)^{\frac{2}{3}} - 5 \times (a^2 - x^2)^{\frac{1}{3}} + 4(a - x)^{\frac{2}{3}} = 0$.

(183.) $\dfrac{\sqrt{a^{\frac{1}{n}} - x^{\frac{1}{n}}}}{x^{\frac{1}{n}}} - \dfrac{\sqrt{a^{\frac{1}{n}} - x^{\frac{1}{n}}}}{a^{\frac{1}{n}}} = \dfrac{x^{\frac{1}{2n}}}{b^{\frac{1}{n}}}$.

(184.) $x - 2\sqrt{x + 2} = 1 + \sqrt[4]{x^3 - 3x + 2}$.

(185.) $\sqrt{x}\sqrt{x^3 - a^3} - 2a^2 = x^2 - 3ax$.

(186.) $x^4 - x^3 + 1 = \dfrac{4x - 5x^2}{4}$. (187.) $\dfrac{(1 + x)^3}{1 + x^3} = \dfrac{1}{a}$.

(188.) $\dfrac{(1 + x)^2}{1 + x^3} + \dfrac{(1 - x)^2}{1 - x^3} = m$.

(189.) $(x + a)\left(1 + \dfrac{1}{x^2 + a^2}\right) + 2\sqrt{ax}\left(1 - \dfrac{1}{x^2 + a^2}\right) = 4a$.

(190.) $\dfrac{1 + x^3}{(1 + x)^3} + \dfrac{1 - x^3}{(1 - x)^3} = a$. (191.) $\dfrac{1 + x^4}{(1 + x)^4} = \frac{1}{2}$.

(192.) $\dfrac{1 + x^4}{(1 + x)^4} = a$. (193.) $\dfrac{(1 + x)^5}{1 + x^5} = \dfrac{1}{a}$.

(194.) $\dfrac{1 + x^4}{\sqrt{1 - x^4}} = \dfrac{2x}{a}$. (195.) $\dfrac{\sqrt{1 + a^2} - a\sqrt{1 + x^2}}{\sqrt{1 + x^2} - x\sqrt{1 + a^2}} = a$.

(196.) $(1 + x)^{\frac{2}{3}} + (1 - x)^{\frac{2}{3}} = (1 - x^2)^{\frac{1}{3}}$.

(197.) $(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{2}{m}} - (a - x)^{\frac{1}{m}} = 6(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{2}{m}}$.

(198.) $16 \times (x^2 + 2)^{\frac{3}{2}} + \dfrac{3}{\sqrt{x^2 + 2}} = 32x^2 + 48$.

(199.) $$\frac{a^4 - \frac{3}{16}b^4}{a^2 + a\sqrt{b^2 + bx} + \frac{1}{2}bx - \frac{1}{4}b^2 - \frac{1}{2}b\sqrt{x^2 - bx + b^2}}$$
$$= a^2 - a\sqrt{b^2 + bx} + \tfrac{1}{2}bx - \tfrac{1}{4}b^2 + \tfrac{1}{2}b\sqrt{x^2 - bx + b^2}.$$

(200.) $(1 + x)\sqrt{1 + a} + (1 - x)\sqrt{1 - a} = 2\sqrt{1 + x^2}$.

(201.) $\dfrac{x}{a + x} + \dfrac{a}{\sqrt{a + x}} = \dfrac{b}{x}$.

(202.) $n\sqrt{a^2 + x^2} - (n - 1)\sqrt{x^2 - 2(n - 1)a^2} = 2n - 1$.

(203.) $\dfrac{x}{2} + 63x^{-\frac{1}{4}} = 220\frac{1}{2} \times x^{-\frac{1}{2}} + 49\sqrt{x} - 1196 = 0$.

(204.) $\dfrac{x^2}{\sqrt{a} + \sqrt{b}} - \sqrt{a} - \sqrt{b})\, x = \left((ab^2)^{-\frac{1}{2}} + (a^2b)^{-\frac{1}{2}}\right)^{-1}$

(205.) $(x + 2)^2 + 2\sqrt{x}(x + 2) - 3\sqrt{x} = 46 + 2x$.

(206.) $\sqrt[4]{a + x} + \sqrt[4]{a - x} = b$.

(207.) $\sqrt{2a^2 - ax - x^2} - (a + x) - \sqrt{ax - x^2} = \sqrt{2ax + x^2}$.

(208.) $x^2 - \dfrac{27x}{4} + 25 = 7x^{\frac{1}{2}}(5 - x)$.

(209.) $4x^2 - 27x + 12x\sqrt{1 + x} = 27$.

(210.) $\dfrac{\sqrt{x + 1} + \sqrt{3x - 1}}{(1 - x)} = \sqrt{a\left(1 + \dfrac{1}{x}\right) - 2}$.

(211.) $a + b\sqrt{x} + x = (b - \sqrt{x})\sqrt{2a + x}$.

(212.) $(x + 2\sqrt{x})^{\frac{1}{3}} - (x - 2\sqrt{x})^{\frac{1}{3}} = 2(x^2 - 4x)^{\frac{1}{6}}$.

(213.) $(x - 1)^2 + (a - 1)^2 - 2(ax + 1) = \sqrt{3(x + a)^2 + 4ax}$.

(214.) $\left(x^{\frac{1}{3}} + \dfrac{1}{x^{\frac{1}{3}}}\right)\sqrt{x^{\frac{1}{3}} + \dfrac{1}{x^{\frac{1}{3}}}} = \sqrt{\dfrac{1}{3}\left(x^{\frac{3}{2}} + \dfrac{1}{x^{\frac{3}{2}}}\right)}$.

EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

IX. (1.) $2x + 3y = 18$, and $3x - 2y = 1$.

(2.) $7(x - 5) = y - 2$, and $4y - 3 = \frac{1}{3}(x + 10)$.

(3.) $\frac{1}{7}(2x - y) = 2y - 3(x + 2)$, and $\frac{1}{3}(y + 3) + \frac{1}{8}(y - x) = 2(x - 4)$.

(4.) $\dfrac{x}{2} - \dfrac{y}{3} = 16$, and $\dfrac{x}{5} - \dfrac{y}{8} = 12$.

(5.) $\dfrac{7x + 6}{11} + y - 16 = \dfrac{5x - 13}{2} - \dfrac{8y - x}{5}$, and

$$3(3x + 4) = 10y - 15.$$

(6.) $9x - 4y = 8$, and $x + \dfrac{7y}{13} = 7\frac{10}{13}$.

(7.) $\dfrac{4}{x} + \dfrac{3}{y} = \dfrac{8}{15}$, and $\dfrac{3}{x} + \dfrac{4}{y} = \dfrac{31}{60}$.

(8.) $(x + 5)(y + 7) = (x - 1)(y - 9) + 104$, and $2x + 10 = 3y + 1$.

(9.) $\dfrac{6x^2 + 130 - 24y^2}{3x + 6y + 1} = 3 - 4y + 2x$, and

$$\frac{12xy - 19x + 137}{4y - 1} = \frac{9xy - 110}{3y - 4}.$$

(10.) $4x + 5y = 40(x - y)$, and $\dfrac{2x - y}{3} + 2y = \dfrac{1}{2}$.

(11.) $\dfrac{1}{3x} + \dfrac{1}{5y} = \dfrac{19}{90}$, and $\dfrac{1}{5x} + \dfrac{1}{3y} = \dfrac{7}{80}$.

(12.) $(x + 5)(y + 7) = (x + 1)(y - 9) + 152$, and $2x + 10 = 3y - 1$.

(13.) $\dfrac{7 + x}{5} - \dfrac{2x - y}{4} = 3y - 12$, and $\dfrac{5y - 7}{2} + \dfrac{4x - 3}{6} = \dfrac{308}{6} - 5x$.

(14.) $\frac{1}{3}(x + y) + \frac{1}{4}(x - y) = 59$, and $5x = 33y$.

(15.) $\dfrac{\dfrac{7x}{4} + 6y}{5} - \dfrac{\dfrac{3y + 6}{5} - \dfrac{3x - 2}{10}}{8} = \dfrac{80 - x}{16}$, and

$$\left(\frac{3x}{2} + \frac{2y}{3} + \frac{5}{2}\right) = 9\left(\frac{x}{2} - \frac{y}{3} + \frac{1}{6}\right).$$

(16.) $\dfrac{7+x}{5} - \dfrac{2x-y}{4} = 3y - 5$, and

$$\frac{5y-7}{2} + \frac{4x-3}{6} = 18 - 5x.$$

(17.) $x - \dfrac{2y-x}{23-x} = 20 - \dfrac{59-2x}{2}$, and

$$y + \frac{y-3}{x-18} = 30 - \frac{73-3y}{3}.$$

(18.) $\dfrac{3x-5y}{3} - \dfrac{2x-8y-9}{12} = \dfrac{y}{2} + \dfrac{1}{8} + \dfrac{1}{4}$, and

$$\frac{x}{7} + \frac{y}{4} + \frac{4}{3} : 4x - \frac{y}{8} - 24 :: 3\tfrac{1}{3} : 3\tfrac{1}{4}.$$

(19.) $2x + .4y = 1.2$, and $3.4x - .02y = .01$.

(20.) $2.4x + 0.32y - \dfrac{0.36x - 0.05}{0.5} = 0.8x + \dfrac{2.6 + 0.005y}{0.25}$, and

$$\frac{0.04y + 0.1}{0.3} = \frac{0.07x - 0.1}{0.6}.$$

(21.) $\dfrac{x}{a} + \dfrac{y}{b} = 1 = \dfrac{x-a}{b} + \dfrac{y-b}{a}$.

(22.) $\dfrac{y}{x} - \dfrac{x}{x+y} = \dfrac{x^2-y^2}{y}$, and $\dfrac{x}{y} - \dfrac{x+y}{x} = \dfrac{y}{x}$.

(23.) $3a^2 + ax = b(b+y)$, and $ax + 2by = a^2$.

(24.) $\dfrac{m}{x} + \dfrac{n}{y} = a$, and $\dfrac{m}{y} + \dfrac{n}{x} = b$.

(25.) $x + y = a$, and $ax = by$.

(26.) $ax + by = m$, and $a'x + b'y = n$.

(27.) $\dfrac{x}{a} + \dfrac{y}{b} = 1 - x$, and $\dfrac{y}{a} + \dfrac{x}{b} = 1 + y$.

(28.) $c(bx + ay) = axy$, and $c(ax - by) = bxy$.

(29.) $(a^2 - b^2)(3x + 5y) = 4\left(a - \dfrac{b}{4}\right)2ab$, and

$$a^2x - \frac{ab^2c}{a+b} + (a+b+c)by = b^2x + (a+2b)ab.$$

(30.) $\dfrac{1}{mx} + \dfrac{1}{ny} = \dfrac{1}{a}$, and $\dfrac{1}{nx} + \dfrac{1}{my} = \dfrac{1}{b}$.

(31.) $ax + by = m$, and $a^2x - b^2y = bm$.

(32.) $a(x^2 + y^2) - b(x^2 - y^2) = 2a$, and $(a^2 - b^2)(x^2 - y^2) = 4ab$.

(33.) $(x + a)(y - b) + 2c = (x - a)(y + b)$, and
$$(x + b)(y - a) = (x + a)(y - b).$$

(34.) $(x + y)a - b(x - y) = 2a^2$, and $(a^2 - b^2)(x - y) = 4a^2b$.

(35.) $(x + y)b = a(x - y)$, and $(x + y)^2 = ax - b(\frac{1}{4}b + y)$.

(36.) $x^2 = ax + by$, and $y^2 = bx + ay$.

(37.) $\dfrac{10x^2 - 12y^2 - 14xy + 2x}{5x + 3y + 3} = \dfrac{4x - 8y + 1}{2}$, and
$$2\sqrt{6 + x} = 3\sqrt{6 - y}.$$

(38.) $x + y = 3y$, and $xy = 18$.

(39.) $x - y = a$, and $y(y + a) + bx = 0$.

(40.) $x - 2y = 2$, and $3xy = 36$.

(41.) $x - y = 2$, and $\dfrac{x}{y} - \dfrac{y}{x} = \dfrac{16}{15}$.

(42.) $5(x - y) = 4y$, and $x^2 + 4y^2 = 181$.

(43.) $x + 4y = 7$, and $x^2 + y^2 = 10$.

(44.) $\dfrac{x + y}{x^2 - y^2} = \frac{1}{4}$, and $xy = 21$.

(45.) $x + y = 1$, and $x^2 - xy = 153$.

(46.) $\dfrac{x^3 + y^3}{x^3 - y^3} = \dfrac{559}{127}$, and $x^2y = 294$.

(47.) $x^2 - xy = y^2 - \dfrac{5xy}{12}$, and $x - 2 = y$.

(48.) $x(x + y) = 66$, and $x^2 - y^2 = 11$.

(49.) $x^2 + y^2 = 5$, and $x - y = 1$.

(50.) $x^2 - xy = 6$, and $x^2 + y^2 = 61$.

(51.) $\dfrac{x^3 - y^3}{x^2y - xy^2} = \dfrac{7}{2}$, and $x + y = 6$.

(52.) $x^3 + y^3 = 35$, and $x + y = 5$.

(53.) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}$, and $\dfrac{2}{xy} = \dfrac{1}{9}$.

(54.) $x^4 - y^4 = 369$, and $x^2 - y^2 = 9$.

(55.) $x + y = 11$, and $x^3 + y^3 = 341$.

(56.) $x - y = 8$, and $x^4 - y^4 = 14560$.

(57.) $x^2 + xy + y^2 = 14x$, and $x^4 + x^2y^2 + y^4 = 84x^2$.

(58.) $x + y = 14$, and $x^4 - y^4 = 14560$.

(59.) $x^4 + y^4 = 641$, and $x^3y + y^3x = 290$.

(60.) $(x^2 + y^2)(x^3 + y^3) = 455$, and $x + y = 5$.

(61.) $x^4 + y^4 = 97$, and $x + y = 5$.

(62.) $x^2 + xy + 4y^2 = 6$, and $3x^2 + 8y^2 = 11$.

(63.) $x^4 + y^4 = 17$, and $x + y = 3$.

(64.) $x^5 + y^5 = 33$, and $x + y = 3$.

(65.) $\dfrac{x^2 + xy + y^2}{x^2 - xy + y^2} = \dfrac{7}{3}$, and $x + y = \dfrac{3xy}{2}$.

(66.) $2x^2 + 3xy + y^2 = 15$, and $5x^2 + 4y^2 = 24$.

(67.) $2x^2 + 3y^2 = 5 = 5(2x + 3y)$.

(68.) $x^4 - x^2 + y^4 - y^2 = 312$, and $x^2 + 2xy + y^2 = 49$.

(69.) $x^3 + y^3 + xy(x + y) = 13$, and $(x^2 + y^2)x^2y^2 = 468$.

(70.) $x^3 + xy^2 = y$, and $y^3 - x = x^2y$.

(71.) $x^2 - 2xy + 3y^2 = 9$, and $x^2 - 4xy + 5y^2 = 5$.

(72.) $x^2 - 2xy - y^2 = 31$, and $x^2 + 4xy - 2y^2 = 202$.

(73.) $x^4 + y^4 - 3x^2y^2 - 2xy = 1$, and $x^3 + y^3 - 2xy^2 - 2y^2 - x = 1$.

(74.) $x^2 + xy + y^2 = 52$, and $xy - x^2 = 8$.

(75.) $x^3 - y^3 = 56$, and $xy(x - y) = 16$.

(76.) $x^2 + y(xy - 1) = 0$, and $y^2 = x(xy + 1)$.

(77.) $x + y = 4$, and $(x^2 + y^2)(x^3 + y^3) = 280$.

(78.) $x + y - z = 6$, $x + z - y = 4$, and $y + z - x = 2$.

(79.) $x + y + z = 29$, $x + 2y + 3z = 62$, and $6x + 4y + 3z = 120$.

(80.) $5x - 11y^{\frac{1}{2}} + 13z^{\frac{1}{3}} = 22$, $4x + 6y^{\frac{1}{2}} + 5z^{\frac{1}{3}} = 31$, and

$x - y^{\frac{1}{2}} + z^{\frac{1}{3}} = 2$.

(81.) $x + \frac{z}{4} = \frac{41}{2}$, $y + \frac{x}{2} = 41$, and $\frac{5y + z}{5} = 34$.

(82.) $\frac{x + y}{44} = \frac{z}{17}$, $\frac{x - y}{20} = \frac{z}{17}$, and $\frac{x^2 - y^2}{3520} = \frac{z}{17}$.

(83.) $x + 2y + 3z = 17$, $2x + 3y + z = 12$, and $3x + y + 2z = 13$.

(84.) $\frac{xy}{x + y} = 1$, $\frac{xz}{x + z} = 2$, and $\frac{yz}{y + z} = 3$.

(85.) $\frac{1}{x} + \frac{1}{y} = \frac{3}{20}$, $\frac{1}{x} + \frac{1}{z} = \frac{7}{60}$, and $\frac{1}{y} + \frac{1}{z} = \frac{1}{6}$.

(86.) $x + y + z = 15$, $x + y - z = 3$, and $x - y + z = 5$.

(87.) $x + y + z = 31$, $x + y - z = 25$, and $x - y - z = 9$.

(88.) $x + y + z = 9$, $x + 3y - 3z = 7$, and $x - 4y + 8z = 8$.

(89.) $x + 2y + 3z = 22$, $2x + z + 3y = 17$, and $3x + y + 2z = 21$.

(90.) $3x + 2y = 32$, $3y + 2z = 25$, and $2x + 3z = 18$.

(91.) $x + 30 = y + z$, $x + 2z = y + 30$, and $3x + y = z + 30$.

(92.) $2x + y = 3y + z = 2x + z = 120$.

(93.) $xy = 3(x + y)$, $xz = 4(x + z)$, and $yz = 6(y + z)$.

(94.) $\frac{2}{x} + \frac{3}{y} - \frac{4}{z} = \frac{1}{12}$, $\frac{3}{x} - \frac{4}{y} + \frac{5}{z} = \frac{19}{24}$, and

$$\frac{4}{x} - \frac{5}{y} + \frac{1}{2} = \frac{6}{z}.$$

(95.) $\frac{2}{x} - \frac{5}{3y} + \frac{1}{z} = 3\frac{4}{27}$, $\frac{1}{4x} + \frac{1}{y} + \frac{2}{z} = 6\frac{11}{72}$, and

$$\frac{5}{6x} - \frac{1}{y} + \frac{4}{z} = 12\frac{1}{36}.$$

(96.) $\frac{1}{x} + \frac{1}{y} = \frac{1}{a}$, $\frac{1}{x} + \frac{1}{z} = \frac{1}{b}$, and $\frac{1}{y} + \frac{1}{z} = \frac{1}{c}$.

(97.) $4x - 5y + mz = 7x - 11y + nz$, and

and $7x - 11y + nz = x + y + pz = 3$.

(98.) $x + y + z = 0$, $(a + b)x + (a + c)y + (b + c)z = 0$,

and $abx + acy + bcz = 1$.

(99.) $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 62$, $\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 47$, and
$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 38$.

(100.) $\frac{3}{x} - \frac{4}{5y} + \frac{1}{z} = 7\frac{2}{5}$, $\frac{1}{2y} - \frac{61}{6} + \frac{z + 6x}{3xz} = 0$, and
$\frac{4}{5x} - \frac{161}{10} = \frac{1}{2y} - \frac{4}{z}$.

(101.) $x^{-2}y^{-1}z = \frac{3}{2}$, $x^{-1}yz^2 = 18$, and $xy^2 = 108z^{-3}$.

(102.) $\frac{105}{xyz} = 1$, $35x = 3yz$, and $7xy = 15z$.

(103.) $x^2y^3 = az^{-4}$, $x^3y^4 = bz^{-2}$, and $x^4y^2 = cz^{-3}$.

(104.) $x + 2y + 3z + 4u = 27$, $3x + 5y + 7z + u = 48$,
$5x + 8y + 10z - 2u = 65$, and $7x + 6y + 5z + 4u = 53$.

(105.) $x + y + z + u = 1$, $16x + 8y + 4z + 2u = 9$,
$81x + 27y + 9z + 3u = 36$, and $256x + 64y + 16z + 4u = 100$.

(106.) $x - 9y + 3z - 10u = 21$, $2x + 7y - z - u = 683$,
$3x + y - 5z + 2u = 325$, and $4x - 6y - 2z - 9u = 516$.

(107.) $xyz = 231$, $xyw = 420$, $yzw = 1540$, and $xzw = 660$.

(108.) $7x - 2z + 3u = 17$, $4y - 2z + t = 11$, $5y - 3x - 2u = 8$,
$4y - 3u + 2t = 9$, and $3z + 8u = 33$.

(109.) $x - ay + a^2z = a^3$, $x - by + b^2z = b^3$, and $x - cy + c^2z = c^3$.

(110.) $bz(a - x) = ay(c - z) = cx(b - y)$, and
$xyz = (a - x)(b - y)(c - z)$.

(111.) $x + y + z = 3m$, $xy + xz + yz = 3m^2$, and $xyz = m^3$.

(112.) $x^2 + xy + y^2 = 37$, $x^2 + xz + z^2 = 28$, and $y^2 + yz + z^2 = 19$.

(113.) $x + y + z = 11$, $x^2 + y^2 + z^2 = 49$, and $yz + 3xy = 3xz$.

(114.) $x + y + z = 10$, $x^2 + y^2 + z^2 = 38$, and $xz = y^2 + 1$.

(115.) $x^2 + xy + y^2 = 13$, $x^2 + xz + z^2 = 31$, and $y^2 + yz + z^2 = 49$.

(116.) $x + y + z = 13$, $x^2 + y^2 + z^2 = 91$, and $y^2 = xz$.

(117.) $xy + z = 5$, $xyz = 4$, and $x^2 - 2y = y^4 - 2xy^2$.

(118.) $x + y + z = 13$, $x^2 + y^2 + z^2 = 61$, and $xy + xz = 2yz$.

(119.) $xy + yz + xz = 11$, $2xy + 3xz + 5yz = 31$, and
$7xy + 5xz + 7yz = 71$.

(120.) $x + y + z = 14$, $x^2 + y^2 + z^2 = 84$, and $xz = y^2$.

(121.) $x + 2y + z = 19$, $x^2 + 4y^2 + z^2 = 133$, and $xz = 4y^2$.

(122.) $x + y + z = \frac{19}{6}$, $x^2 + y^2 + z^2 = \frac{133}{36}$, and $xz = y^2$.

(123.) $xz = y^2$, $(x + y)^2 = z(x + y) - 3$, and
$(x + y + z)(z - x - y) = 7$.

(124.) $x + y + z = m$, $ax + by + cz = 0$, and $a^2x + b^2y + c^2z = 0$.

(125.) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, and
$\frac{1}{x} + \frac{1}{a} = \frac{1}{y} + \frac{1}{b} = \frac{1}{z} + \frac{1}{c}$.

(126.) $ay + bx = c$, $cy + bz = a$, and $az + cx = b$.

(127.) $xy = a(x + y)$, $xz = b(x + z)$, and $yz = c(y + z)$.

(128.) $\frac{xy + xz - yz}{a^2} = \frac{xy + yz - xz}{b^2} = \frac{xz + yz - xy}{c^2} = 1$.

(129.) $x(x + y + z) = a^2$, $y(x + y + z) = b^2$, and $z(x + y + z) = c^2$.

(130.) $x + y + z = a$, $x^2 + y^2 + z^2 = a^2$, and $xy + zy = 2xz$.

(131.) $4xy(y + x - z) = 3$, $2yz(z + y - x) = 15$, and
$zx(x + z - y) = 3$.

(132.) $\frac{a^3x}{y^2z^2} = \frac{b^3y}{z^2x^2} = \frac{c^3z}{x^2y^2} = 1$.

(133.) $\left(\frac{x}{y}\right)^{\frac{1}{2}} + \left(\frac{y}{x}\right)^{\frac{1}{2}} = \frac{61}{\sqrt{xy}} + 1$, and $\sqrt[4]{x y} + \sqrt[4]{xy^3} = 78$.

(134.) $x^2 - 6\sqrt{x^2y} = 27$, and $x - 2\sqrt{xy} = 3$.

(135.) $\frac{x - \sqrt{x^2 - y^2}}{x + \sqrt{x^2 - y^2}} = x$, and $\frac{x}{y} = \sqrt{\frac{1 + x}{1 - y}}$.

(136.) $\sqrt{ax} + \sqrt{by} = \frac{1}{2}(x + y)$, and $(x + y) = 2(a + b)$.

(137.) $(x + y)^2 = x^4 + x^2y^2 + y^4$, and $x^4 + 4x^2y = 8xy^3 - 4y^4$.

(138.) $\sqrt{y} - \sqrt{a-x} = \sqrt{y-x}$, and

$\sqrt{y-x} + \sqrt{a-x} : \sqrt{a-x} :: 5 : 2.$

(139.) $yx + y\sqrt{x^2-y^2} = a(\sqrt{x+y} + \sqrt{x-y})$, and

$\sqrt[4]{x+y} + \sqrt[4]{x-y} = y.$

(140.) $x^4+y^4-1 = 2xy+3x^2y^2$, and $x^3+y^3-1 = x+2y^2+2xy^2$.

(141.) $x^{\frac{1}{4}} + y^{\frac{1}{3}} = 6$, and $x^{\frac{3}{4}} + y^{\frac{2}{3}} = 126$.

(142.) $x^2y + y^3 = 116$, and $xy^{\frac{1}{2}} + y = 14$.

(143.) $\sqrt[3]{x} + \sqrt[3]{y} = 6$, and $x + y = 72$.

(144.) $\sqrt[3]{x} + \sqrt[3]{y} = 3$, and $x + y = 9$.

(145.) $\sqrt{4y-x} + \sqrt{y-x} = 2\sqrt{2y-x}$, and

$3\sqrt{x^2-6y} + 4\sqrt{y^2-9x} : 4\sqrt{x^2-6y} :: 7 : 4.$

(146.) $\dfrac{\sqrt[3]{x+y}}{8y} + \dfrac{\sqrt[3]{x+y}}{8x} = \dfrac{8}{63}$, and

$\dfrac{\sqrt[3]{x-y}}{y} + \dfrac{\sqrt[3]{x-y}}{x} = \dfrac{32}{63}.$

(147.) $\frac{1}{2}(x^2-y^2) = 17ay + 3y^2 + 4a\sqrt{2ay-y^2}$, and

$x^2 + y^2 = 2ay + 8\sqrt{y}(x\sqrt{a} - y\sqrt{y}).$

(148.) $x^{\frac{4}{3}} + y^{\frac{2}{3}} = 20$, and $x^{\frac{2}{3}} + y^{\frac{1}{3}} = 6$.

(149.) $(x^2+y^2)^2 + 4xy(x+y)^2 = 1396$, and $x - y = 4$.

(150.) $\dfrac{\sqrt{x} + \sqrt{x-y}}{\sqrt{x} - \sqrt{x-y}} = 4$, and $y = 4\sqrt{x}$.

(151.) $x + y = a$, and $x^4 + y^4 = b^4$.

(152.) $x^4y^3 - x^3y^4 = 216$, and $xy(x-y) = 6$.

(153.) $x^2y^2 + 4xy = 96$, and $x + y = 6$.

(154.) $x(x + \sqrt[3]{xy^2}) = 208$, and $y(y + \sqrt[3]{x^2y}) = 1053$.

(155.) $x^2 + xy = 6$, and $xy - y^2 = 1$.

(156.) $x^2+y^2+xy(x+y) = 68$, and $x^2(x-3) + y^2(y-3) = 12$.

(157.) $x^2 + y^2 - (x+y) = 18$, and $x + y = 19 - xy$.

(158.) $\frac{x^2 + y^2}{10} = \frac{x + y}{3}$, and $xy = 8$.

(159.) $xy(x + y) = 84$, and $x^2 + y^2 = 3600\,x^{-2}y^{-2}$.

(160.) $x^2 + y^2 = c^2$, and $n(x + y) = m(x - y)$.

(161.) $y(x - \sqrt{x^2 - y^2}) = a(\sqrt{x + y}) + \sqrt{x - y}$, and

$(x + y)^{\frac{3}{2}} - (x - y)^{\frac{3}{2}} = b$.

(162.) $x^2 + xy + y^2 = 7(x + y)$, and $(x^2 - xy + y^2) = 9(x - y)$.

(163.) $\frac{xy}{x + y} = a$, and $\frac{x^2y^2}{x^2 + y^2} = b^2$.

(164.) $\frac{1}{x} + \frac{1}{y} = m$, and $\frac{1}{x^2} + \frac{1}{y^2} = n^2$.

(165.) $\frac{1+x}{1+y} + \frac{1-y}{1-x} = \frac{4}{13}$, and $\frac{1+x}{1-y} + \frac{1+y}{1-x} = \frac{9}{13}$.

(166.) $\frac{x + y - \sqrt{x^2 + y^2}}{x + y + \sqrt{x^2 + y^2}} = \frac{2x}{a}$, and $x\sqrt{a - y} = y\sqrt{a + x}$.

(167.) $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} = \frac{27}{4}$, and $x - y = 2$.

(168.) $\frac{\sqrt{3x}}{\sqrt{x + y}} - 2 + \frac{\sqrt{x + y}}{\sqrt{3x}} = 0$, and $xy - 54 = x + y$.

(169.) $\sqrt{x} - \sqrt{y} = \sqrt{y + 2}$, and $\frac{x + 8}{8} = \sqrt{y + 2}$.

(170.) $3x^2 + 4y^2 = 7xy$, and $x^{\frac{3}{2}} - yx^{\frac{1}{2}} = \frac{2y^2}{9}$.

(171.) $5ax + 12y(a - x) = 0$, and $x^2 + a^2 = y^2$.

(172.) $x^2y - 4 = 4y\sqrt{x} - \frac{y^3}{4}$, and $(\sqrt{x} - \sqrt{y})\sqrt{xy} = x^{\frac{3}{2}} - 8$.

(173.) $x^2 + xy + y^2 = a^2$, and $x + \sqrt{xy} + y = b$.

(174.) $(x^2 + y^2)(x^3 + y^3) = 455$, and $x + y = 5$.

(175.) $3(x^2 + y^2)y = 26x$, and $2x(x^2 - y^2) = 15y$.

(176.) $\sqrt{x} - \sqrt{y} = x + \sqrt{xy}$, and $(x + y)^2 = 2(x - y)^2$.

(177.) $(x - 2)y - 2y^2 + x = \sqrt{xy}(y^2 - 1)$, and

$xy(xy - 18) = 4(\sqrt{xy} - 12)$.

(178.) $\frac{x}{y} - \frac{y}{x} = \frac{x+y}{x^2+y^2}$, and $\frac{x^2}{y^2} - \frac{y^2}{x^2} = \frac{x-y}{y^2}$.

(179.) $\frac{\sqrt{x^2+y^2} + \sqrt{x^2-y^2}}{\sqrt{x^2+y^2} - \sqrt{x^2-y^2}} = \frac{5+\sqrt{7}}{5-\sqrt{7}}$, and $\frac{8y}{3} - 12 = \frac{x^2}{16}$.

(180.) $(\sqrt{x} - 3\sqrt{y})^2 + 5 - 2\sqrt{y+2} = \frac{9x^2}{64}$, and

$7 - 10\sqrt{xy} = y(x-16)$.

(181.) $a(1-xy) = x\sqrt{1-y^2}$, and $\sqrt{x}(1-xy) = y - x$.

(182.) $x^n + y^n = a^n$, and $xy = b^2$.

(183.) $x + \sqrt{3y^2 - 11 + 2x} = 7 + 2y - y^2$, and

$\sqrt{3y - x + 7} = \frac{x+y}{x-y}$.

(184.) $x^3 - y^3 : (x-y)^3 :: 61 : 1$, and $xy = 320$.

(185.) $2(x^2+y^2)(x+y) = 15xy$, and $4(x^4-y^4)(x^2+y^2) = 75x^2y^2$.

(186.) $(x^2-y^2)(x-y) = 3xy$, and $(x^4-y^4)(x^2-y^2) = 45x^2y^2$.

(187.) $\sqrt{y} + \sqrt{x} : \sqrt{y} - \sqrt{x} :: \sqrt{x} + 2 : 1$, and

$\sqrt{y} + 2 - \sqrt{x} : \sqrt{x} :: 3x + \sqrt{x} + \sqrt{y} : \sqrt{xy}$.

(188.) $x - 8y\sqrt{x^2 - 9xy^2} = (9 - 16x)y^2$, and $5x - 4 = 25y^2$.

(189.) $\sqrt{5\sqrt{x} + 5\sqrt{y}} = 10 - (\sqrt{x} + \sqrt{y})$, and $x^{\frac{5}{2}} + y^{\frac{5}{2}} = 275$.

(190.) $2\sqrt{6\sqrt{x} + 6\sqrt{y}} + \sqrt{x} = 18 - \sqrt{y}$, and $x - y = 12$.

(191.) $\sqrt{(1+x)^2+y^2} + \sqrt{(1-x)^2+y^2} = 4$, and $(4-x^2)^2 + 4y^2 = 18$.

(192.) $\frac{x + \sqrt{x^2-y^2}}{x - \sqrt{x^2-y^2}} + \frac{x - \sqrt{x^2-y^2}}{x + \sqrt{x^2-y^2}} = 4\frac{1}{4}$, and

$x(x+y) + \sqrt{x^2 + xy + 4} = 52$.

(193.) $\sqrt{\frac{x+y^2}{4x}} + \frac{y}{\sqrt{y^2+x}} = \frac{y^2}{4}\sqrt{\frac{4x}{y^2+x}}$, and

$\sqrt{x} + \sqrt{x-y-1} = (y+1)(\sqrt{x} - \sqrt{x-y-1})$.

(194.) $(x-2)y - \sqrt{xy}(y^2-1) = 2y^2 - x$, and

$xy(xy-18) = 4\sqrt{xy} - 48$.

(195.) $xy + xy^2 = 12$, and $x + xy^3 = 18$.

(196.) $\dfrac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}} = \dfrac{17}{4} - \dfrac{x - \sqrt{x^2 - y^2}}{x + \sqrt{x^2 - y^2}}$, and

$$2x^2 + 6xy - \sqrt{x^2 + 3xy - 21} = 162.$$

(197.) $2x + y = 26 - \sqrt{2x + y + 4}$, and

$$\frac{2x + \sqrt{y}}{2x - \sqrt{y}} = \frac{16}{15} + \frac{2x - \sqrt{y}}{2x + \sqrt{y}}.$$

(198.) $\dfrac{x - \sqrt{x^2 - y^2}}{x + \sqrt{x^2 - y^2}} = x$, and $\dfrac{x}{y} = \sqrt{\dfrac{5}{2}\left(\dfrac{1 - x^2}{1 - y^2}\right)}$.

(199.) $\sqrt{x} + \sqrt{y} = 4(\sqrt{x} - \sqrt{y})$, and $x^2 - y^2 = 544$.

(200.) $\dfrac{2xy + y\sqrt{x^2 - y^2}}{14} = \sqrt{\dfrac{x + y}{2}} + \sqrt{\dfrac{x - y}{2}}$, and

$$\left(\frac{x + y}{2}\right)^{\frac{3}{2}} + \left(\frac{x - y}{2}\right)^{\frac{3}{2}} = 9.$$

(201.) $\sqrt{x} - \sqrt{y} = 21$, and $\sqrt[4]{x} + \sqrt[4]{y} = 7$.

(202.) $xy + \sqrt{x^2y^2 - y^4} = 8(\sqrt{x + y} + \sqrt{x - y})$, and

$$(x + y)^{\frac{3}{2}} - (x - y)^{\frac{3}{2}} = 26.$$

(203.) $\sqrt{x + y} - \sqrt{x - y} = a$, and $\sqrt[4]{x + y} + \sqrt[4]{x - y} = b$.

(204.) $3\sqrt{x + y} = 4 + 4(x + y)^{-\frac{1}{2}}$, and $\sqrt{x+y} + \sqrt{x-y} = 5$.

(205.) $3x^2 + 4y^2 = 7xy$, and $x^{\frac{3}{2}} - \frac{2}{3}y^2 = yx^{\frac{1}{2}}$.

(206.) $a^2x^{2n} - b^2y^{2n} = c^2 - d^2$, and $ax^n + by^n = c + d$.

(207.) $x^3 + x\sqrt[3]{xy^3} = 208$, and $y^3 + y\sqrt[3]{x^2y} = 1053$.

(208.) $2\sqrt{x^2 - y^2} + x + y = 2(x - 1)$, and $15(x^2 + y^2) = 34xy$.

(209.) $\sqrt{x} - \sqrt{y} = x + \sqrt{xy}$, and $(x + y)^2 = 2(x - y)^3$.

(210.) $(x + y)^m - (x + y)^{-m} = a$, and $x - y = b$.

(211.) $x^m a^n + y^n b^m = 2\sqrt{(ax)^m (by)^n}$, and $xy = ab$.

(212.) $x^m y^n = (ac)^m b^n$, and $x^n y^m = (bc)^n a^m$.

(213.) $x^2 y - 4y\sqrt{x} = 4 - \frac{1}{4}y^2$, and $x^{\frac{3}{2}} - \sqrt{xy}\,(\sqrt{x} - \sqrt{y}) = 3$.

(214.) $\dfrac{x^2}{y^2} + \dfrac{2x + y}{\sqrt{y}} = 20 - \dfrac{y^2 + x}{y}$, and $4y - x = 8$.

(215.) $x + y = a$, and $x^4 + y^4 = d^4$.

(216.) $(\sqrt{x} + \sqrt{y})^2 + (\sqrt[4]{x} + \sqrt[4]{y})^2 = 210 + \sqrt[4]{xy}\,(\sqrt[4]{xy} + 1)$, and
$(\sqrt{x} - \sqrt{y})^2 + (\sqrt[4]{x} - \sqrt[4]{y})^2 = 126 - \sqrt[4]{xy}\,(\sqrt[4]{xy} + 1)$.

(217.) $\dfrac{x + y + \sqrt{x^2 - y^2}}{x + y - \sqrt{x^2 - y^2}} = \dfrac{9}{8y}(x + y)$, and
$(x^2 + y)^2 + x - y = 2x(x^2 + y) + 506$.

(218.) $\dfrac{x + y - \sqrt{x^2 + y^2}}{x + y + \sqrt{x^2 + y^2}} = \dfrac{x}{a}$, and $\dfrac{x}{y} = \sqrt{\dfrac{a + x}{a - y}}$.

(219.) $\dfrac{1}{x}\left(1 - \dfrac{4}{y}\right) + \dfrac{1}{y}\left(1 - \dfrac{1}{x}\right) = 1$, and
$x + y + 12 = 2xy$.

(220.) $x^4 + y^4 - 3x^2y^2 - 2xy = 1$, and $x^3 + y^3 - 2y^2x - 2y^2 = x + 1$.

(221.) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{m}$, and $\dfrac{1}{x^3} + \dfrac{1}{y^3} = \dfrac{1}{n^3}$.

(222.) $x^4 = mx + ny$, and $y^4 = my + nx$.

(223.) $x^2 - y^2 = a^2$, and $(x + y + b)^2 + (x - y + b)^2 = c^2$.

(224.) $a(1 - xy) = x\sqrt{1 - y^2}$, and $\sqrt{x}\,(1 - xy) + x = y$.

(225.) $\dfrac{y - x + \sqrt{2xy - 3x^2}}{3(y - 2x)} = \dfrac{(2y - 3x)^{\frac{3}{2}} + x^{\frac{3}{2}}}{(2y - 3x)^{\frac{3}{2}} - x^{\frac{3}{2}}}$, and
$\dfrac{y}{x^2} + \dfrac{16}{81}(x - \sqrt{x} - \tfrac{3}{4}) = \dfrac{4}{9x}(2\sqrt{xy} - \sqrt{y})$.

(226.) $(x^6 + 1)y = (y^2 + 1)x^3$, and $(y^6 + 1)x = 9y^3(x^2 + 1)$.

(227.) $x^2 + 6x\sqrt[3]{y^2} + \sqrt[3]{y^4} = 128$, and $\sqrt{x^3}\,\sqrt[3]{y} + y\sqrt{x} = 32$.

(228.) $x^5 + y^5 = (x + y)^3 \times xy$, and $y^2\sqrt{x} = (x + y)^{\frac{3}{2}}$.

(229.) $x^2y^2 + xy^4 = 156$, and $2x^3y^2 - x^2y^3 = 144$.

(230.) $(x^2 + y^2)xy = 78$, and $x^4 + y^4 = 97$.

(231.) $x^4 + y^4 = 17$, and $2xy(x^2 + y^2) + 3x^2y^2 = 32$.

(232.) $(x + y)(x^3 + y^3) = 76$, and $(x + y)^3 = 64(x - y)$.

(233.) $5 - 2\sqrt{y + 2} = \dfrac{9x^2}{64} - (\sqrt{x} - 3\sqrt{y})^2$, and

$$\frac{7}{y} - 10\sqrt{\frac{x}{y}} = x - 16.$$

(234.) $3x + \frac{2}{3}\sqrt{xy^3 + 9x^2y} = (x - \frac{1}{3})y$, and $6x + y : y :: x + 5 : 3$.

(235.) $3x - x\sqrt{\dfrac{5x^2}{4} - 2y + 8} = 2 - y$, and

$$\frac{\sqrt{x + y}}{2x} - \frac{3x}{4} = \frac{2x - 3}{\sqrt{x + y}} - \frac{3y}{2x}.$$

(236.) $1 + \dfrac{x^2}{5y^2} + \dfrac{x^4}{25y^4} + \text{\&c. to inf.} = \dfrac{5x^2}{y^2}$, and

$$1 + \frac{1}{2}(x + y)^{-1} + \frac{1.3}{2.4}(x + y)^{-2} + \text{\&c.} = \sqrt{1{\cdot}25}.$$

(237.) $x^2 + xy + y^2 = a^2$, and $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y = b$.

(238.) $xy - \dfrac{1200}{x} = 252\left(\dfrac{y}{x}\right)^{\frac{1}{3}} - (xy^2)^{\frac{1}{3}}$, and

$$2x^{\frac{4}{3}} + 16y^{\frac{4}{3}} + 42(xy)^{\frac{2}{3}} = 4\sqrt[3]{xy}\,(5x^{\frac{2}{3}} + 11y^{\frac{2}{3}}).$$

(239.) $\dfrac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}(x^{\frac{2}{3}} - 1) + \dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(2x^{\frac{2}{3}} - 1) = \dfrac{4y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(y^{\frac{1}{3}} + x^{\frac{1}{3}}) + \dfrac{3y}{x^{\frac{1}{3}}} + 2$,

$$\text{and } \frac{x^{\frac{4}{3}}}{y^{\frac{4}{3}}} - \frac{2x^{\frac{2}{3}}}{y} - \frac{2x^{\frac{1}{3}}}{y^{\frac{1}{3}}} = \frac{133}{36} \times \frac{1}{y^{\frac{2}{3}}} - \frac{2}{x^{\frac{1}{3}}} - \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}.$$

(240.) $xy = a$, $xz = b$, $xu = c$, and $xyzu = d$.

(241.) $\sqrt[3]{\left\{\dfrac{27y^{\frac{3}{2}} - 1}{x^3 + 8y^3 - 2xy^{\frac{3}{2}}}\right\}} = 3\sqrt{\dfrac{x}{y}}$, and

$$3x^2 + 42xy + 16y^2 = 4\sqrt{xy}\,(5x + 11y).$$

(242.) $2(x^2 + y^2) = 2x + a$, and $x^3(x - 2) + 6y^2x(x-1) + y^4 = \dfrac{b - 2x}{2}$.

(243.) $(1 - x^2)^2(1 + y^2) - (1 + x^2)^2(1 - y^2) = 4x^2\sqrt{1 + y^4}$, and
$$\frac{4xy}{1 - y^2} = \sqrt{2}(1 - x^2).$$

(244.) $m^{x^2} \cdot n^{z^2} = a$, and $x : z :: r : s$.

(245.) $(2 + 4xy - 3x^2)^2 = 2 - 4x^2y^2 + 3x^4$, and
$$(x^2 - 1)^2 = (2y^2 + x^2 + 1)(2y^2 - x^2 - 1).$$

(246.) $\dfrac{x^3}{y} - \dfrac{9}{x} = \dfrac{8}{x^{\frac{3}{2}}}\sqrt{\dfrac{x^5}{y}}$, and $\dfrac{y}{x^2} + \dfrac{x}{y\sqrt{x}} \cdot \sqrt{\dfrac{y}{x}} = \dfrac{10x^2}{81y^2}$.

(247.) $\left(\sqrt{\dfrac{x}{y}} - \sqrt{\dfrac{y}{x}}\right)^2 + \sqrt{y}\left(\sqrt{x} - \dfrac{\sqrt{y}}{2}\right) =$
$$\frac{x}{\sqrt{y}}\left(\sqrt{x} - \frac{y}{\sqrt{2}}\right), \text{ and}$$
$$9\sqrt{\frac{x}{y}} + 3\sqrt{\frac{y}{x}} = \frac{21\sqrt{2x} - 1}{2}\sqrt{\frac{y}{x}} + \frac{1}{2\sqrt{xy}}.$$

(248.) $\dfrac{x^2y^2}{2} + 4 - 40y^2 = 140 - y^2\sqrt{x^2 - \dfrac{272}{y^2}}$, and
$$x^2 - \frac{2}{y}\left(15x + \frac{3}{y}\right) = \frac{30}{y^2} + \frac{5x}{y}.$$

(249.) $x^2(b - y) = (y - n)ay$, and $y^2(a - x) = (x - n)bx$.

(250.) $y^4 = x^2(ay - bx)$, and $x^2 = ax - by$.

(251.) $\dfrac{3 + 2x^2 - 4x^4}{x^2 - 1} = y^2(1 - 2y^2)$, and $(2x^2 - 1)(2y^2 - 1) = 3$.

(252.) $x^5 + y^5 = xy(x + y)^3$ and $y^2\sqrt{x} = (x + y)^{\frac{3}{2}}$.

(253.) $a(\sqrt{x + y} + \sqrt{x - y}) = xy - y\sqrt{x^2 - y^2}$, and
$$\sqrt[4]{x + y} + \sqrt[4]{x - y} = b.$$

(254.) $2 + 4xy - 3x^2 = \sqrt{2 - 4x^2y^2 + 3x^4}$, and
$$5x^2y^2 + \frac{27x^4}{32} = \frac{9x^3y}{2} + 2xy + 1.$$

(255.) $5y + \dfrac{\sqrt{x^2 - 15y - 14}}{5} = \dfrac{x^2}{3} - 36$, and

$$\frac{x^2}{8y} + \frac{2x}{3} = \sqrt{\frac{x^3}{8y} + \frac{x^2}{4}} - \frac{y}{2}.$$

(256.) $(x - 2)y + x - 2y^2 = \sqrt{xy}\,(y^2 - 1)$, and

$$xy\,(xy - 18) = 4(\sqrt{xy} - 12).$$

(257.) $(xy^2 + x)^{\frac{1}{2}} + x^{\frac{1}{2}} = y\,(x + 9)^{\frac{1}{2}} + 3y$, and

$$x\,(y + 1)^2 = 4(9y^3 + 16).$$

(258.) $\sqrt[3]{x+y} + \sqrt[3]{x-y} = a^{\frac{1}{3}}$, and $(x^2+y^2)^{\frac{1}{3}} + (x^2-y^2)^{\frac{1}{3}} = a^{\frac{2}{3}}$.

(259.) $(x^2 + y^2 + c^2)^{\frac{1}{3}} + (x - y + c)^{\frac{2}{3}} = (32xy)^{\frac{1}{3}}$, and

$$xy = c\,(x - y).$$

(260.) $\dfrac{y}{x}\sqrt{\dfrac{x}{y}} + \dfrac{1}{2}\sqrt{\dfrac{x}{y}} \times \sqrt[4]{\dfrac{y^3}{x^3}} = 5$, and

$$\frac{2x^2}{y} - \frac{x}{3\sqrt{y}} = \frac{1}{3}.$$

(261.) $\sqrt{x^2 + \sqrt[3]{x^4y^2}} + \sqrt{y^2 + \sqrt[3]{x^2y^4}} = a$, and $x + y + 3\sqrt[3]{bxy} = b$.

(262.) $6x - x\sqrt{5x^2 - 8(y - 4)} = 2(2 - y)$, and

$$\frac{\sqrt{x + y}}{2x} - \frac{3x}{4} = \frac{2x - 3}{\sqrt{x + y}} - \frac{3y}{2x}.$$

(263.) $\sqrt{\dfrac{x}{6} - 18 + 7y^4} - \dfrac{11}{13}\sqrt{9y^2 - x} = \dfrac{x}{120} + \dfrac{41}{10} + \dfrac{7y^4}{20}$,

and $y\sqrt{x + 9} - \sqrt{xy^2 + x} = x^{\frac{1}{2}} - 3y$.

(264.) $\sqrt{\left\{\dfrac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{4}{3}}y^{\frac{4}{3}}}\right\}} + 40\sqrt{\left\{\dfrac{x^{\frac{2}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}}\right\}} = 241$, and

$$\left[1 + \left(\frac{y}{x}\right)^{\frac{2}{3}}\right] \times \left[3x^{\frac{4}{3}}y^{\frac{2}{3}} + \frac{91}{216}\sqrt{x^2 + x^{\frac{4}{3}}y^{\frac{2}{3}}}\right] = \frac{125 - 216(x^2 + y^2)}{216}.$$

X.

PROBLEMS PRODUCING SIMPLE EQUATIONS.

(1.) What number is that, from the treble of which if 48 be subtracted, the remainder is 42?

(2.) To determine two numbers, such that their difference may be 5, and the difference of their squares 75.

(3.) Find a number such that its third part being added to it, the sum is less than 9 by as much as the number itself is greater than 5.

(4.) What number is that, the double of which exceeds four-fifths of its half by 40?

(5.) Find two consecutive numbers such that the half and fifth part of the less may together be equal to the sum of the third and fourth parts of the greater.

(6.) To find two numbers with these conditions, viz. that half the first with a third part of the second may make 9, and that a fourth part of the first with a fifth part of the second may make 5.

(7.) Of £3200., A. has £400. more than B., and B. has £200. more than C.: find the share of each.

(8.) Find two numbers in the proportion of 9 : 7, such that the square of their sum shall be equal to the cube of their difference.

(9.) Required the number of which $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ together are as much greater than 223, as $\frac{1}{5}$, $\frac{1}{6}$ and $\frac{1}{7}$ of it together are less than the same.

(10.) To divide the number 2 into two such parts, that a third of the one part added to a fifth of the other may make $\frac{3}{5}$.

(11.) A. and B. began to play with equal sums; A. won £1. 10*s.*, and then their money was in the proportion of 13 : 7. How much had each when they left off playing?

(12.) The ages of two brothers differ by 20 years, and one is as much above 25 as the other is below 25; what are their ages?

(13.) Find a number of which the cube root is $\frac{1}{5}$ the square root.

(14.) Find a fraction which becomes $\frac{4}{5}$ when unity is added to its numerator, and $\frac{1}{2}$ if unity be added to its denominator.

(15.) To find three numbers, such that the sum of the first and second shall be 7, the sum of the first and third 8, and the sum of the second and third 9.

(16.) Divide £64. among 3 persons, so that the first may have 3 times as much as the second; and the third, one third as much as the first and second together.

(17.) A person, dying, bequeathed his fortune, which was 2800*l.*, to his son and daughter, in this manner; that for every half-crown the son might have, the daughter was to have a shilling. What, then, were their two shares?

(18.) A gamester at one sitting lost $\frac{1}{3}$ of his money, and then won 10*s.*; at a second, he lost $\frac{1}{3}$ of the remainder, and then won 3*s.*; after which he had 3 guineas left. How much money had he at first?

(19.) Three persons, A., B., C., make a joint contribution, which in the whole amounts to 400*l.*; of which sum B. contributes twice as much as A. and 20*l.* more; and C. as much as A. and B. together. What sum did each contribute?

(20.) A person being asked the hour of the day, answered thus:— If $\frac{3}{8}$ of the number of hours remaining till midnight be multiplied by 4, the product will as much exceed 12 hours, as $\frac{1}{2}$ of the present hour from noon is below 4. What was the hour after noon?

(21.) Two coaches start at the same time from York and London, a distance of 200 miles: the one from London travels at $9\frac{1}{4}$ miles an hour, and that from York at $10\frac{3}{4}$. Where will they meet, and in what time from starting?

(22.) A person paid a bill of 100*l.* with half-guineas and crowns, using in all 202 pieces; how many pieces were there of each sort?

(23.) A. and B. begin to play with equal sums; A. won 5*l.*, and then three times A.'s money was equal to eleven times B.'s. What had each at first?

(24.) Says A. to B., If you give me 10 guineas of your money, I shall then have twice as much as you will have left. But says B. to A., Give me 10 of your guineas, and then I shall have three times as many as you. How many had each?

(25.) A messenger starts on an errand at the rate of 4 miles an hour; another is sent an hour and a half after to overtake him; the latter walks at the rate of $4\frac{3}{4}$ miles an hour: when will he overtake the former?

(26.) A garrison of 500 men was victualled for 48 days; after 15 days it was reinforced, and then the provisions were exhausted in 11 days. Required the number of men in the reinforcement.

(27.) A. and B. together possess 150*l.* and C. has 50*l.* more than D.; also A. has twice as much as C., and B. thrice as much as D. Required the money of each.

(28.) What number is that, of which the half, the fifth, and the seventh parts are together equal to 59?

(29.) In the election of a Member of Parliament, $\frac{1}{16}$ of the constituency refuse to vote, and of two candidates the one who is supported by $\frac{19}{40}$ of the whole constituency is returned by a majority of 5: find the number for each candidate.

(30.) Required a number from which if 84 be taken, three times the remainder will exceed the required number by a fourth of itself.

(31.) A person spends 2*s*. at a tavern: he then borrows as much money as he has left, and spends 2*s*. at another tavern; borrowing again as much as was left, he spends 2*s*. at a third tavern; and repeating this, he spends 2*s*., all he now has, at a fourth tavern: what had he at first?

(32.) The greater of two numbers is equal to four times the less, and the excess of the greater over the less is 24; required the two numbers.

(33.) A person pays an income tax of 7*d*. in the pound, and a poor rate exceeding it by 22*l*. 10*s*., and has 486*l*. left: find his income.

(34.) Required a number such, that if it be multiplied by 11, and 320 be taken from the product, the tenth part of the remainder will be 20 less than the number itself.

(35.) There are two kinds of coin, of which a and b pieces respectively are equivalent to 1*l*.: how many pieces of each kind must be taken so that c pieces together may be equivalent to 1*l*.?

(36.) If A. and B. have between them 1200*l*., A. and C. 1400*l*., B. and C. 1500*l*.; how much has each?

(37.) A tradesman, after expending 100*l*. a year, augments the remainder of his property by one third part of it, and at the end of 3 years his original property is doubled: what had he at first?

(38.) A common of 864 acres is to be divided among three landowners, A., B., C., so that A.'s share shall be to B.'s as 5 to 11, and that C. shall receive as much as A. and B. together; required the share of each.

(39.) Divide 30 into two such parts, that one may be the square of the other.

(40.) A farmer engaged a labourer on condition of paying him 1*s*. 4*d*. a day for every day he should work, and of charging him 9*d*. for his board every day he should be idle. Now, at the end of a year (313 days) the man was entitled to 19*l*. 10*s*. 3*d*. How many days did he work?

(41.) Find two numbers, the sum of whose square is 100, and their product 48.

(42.) Find two numbers whose product is equal to the difference of their squares, and the sum of their squares to the difference of their cubes.

(43.) A general after detatching $\frac{5}{13}$ of his army to take possession of a height, and $\frac{5}{13}$ of the remainder to reconnoitre the enemy, had 1280 men left; what was his whole force?

(44.) Divide 10 into three such parts, that when the first is multiplied by 2, the second by 3, and the third by 4, the three products may be equal.

(45.) Let 10 be divided into 4 parts, such that when they are respectively divided by 2, 3, 4, and 5, the quotients will be in the same proportion as 6, 7, 8, and 9.

(46.) Find the fraction which, if 1 be added to its numerator, becomes $\frac{1}{3}$, but if 1 be added to its denominator, becomes $\frac{1}{4}$.

(47.) A person distributed p shillings among n persons, giving 9*d.* to some, and 15*d.* to the rest; how many were there of each?

(48.) A cask, which held 60 gallons, was filled with a mixture of brandy, wine, and cyder, in this manner, viz. the cyder was 6 gallons more than the brandy, and the wine was as much as the cyder and $\frac{1}{3}$ of the brandy; how much was there of each?

(49.) A cistern is filled in 24 minutes by 3 pipes, one conveying 8 gallons more, and another 7 gallons less, than the third, every 3 minutes; the cistern holds 1088 gallons: how much flows through each pipe in a minute?

(50.) A farmer buys a sheep for £P., and sells b of them at a gain of 5 per cent.: at what price ought he to sell the remainder to gain 10 per cent. on the whole?

(51.) A general, disposing his army into a square form, finds that he has 284 men more than a perfect square; but increasing the side by 1 man, he then wants 25 men to complete the square; how many men had he under his command?

(52.) A boy at a fair spends his money in oranges; if he had received 5 more for his money, they would have averaged a half-penny each less, if 3 less, a half-penny each more: how much did he spend?

(53.) The stock of three traders amounted to 760*l.*; the shares of the first and second exceeded that of the third by 240*l.* and the sum of the second and third exceeded the first by 360*l.*; what was the share of each?

(54.) A. sets out from C. to go to D., at the same time that B. sets out from D. to go to C.; A. arrives at D. a hours, and B. at C. b hours after they meet: in what time did each perform the journey?

(55.) A garrison of 1000 men was victualled for 30 days, after 10 days it was reinforced, and then the provisions were exhausted in 5 days; required the number of men in the reinforcement.

(56.) What are those two numbers, of which the greater is to the less as their sum is to 20, and as their difference is to 10?

(57.) To find three numbers in arithmetical progression, of which the first is to the third as 5 to 9, and the sum of all three is 63.

(58.) There was a run, during the late panic, on two bankers, A. and B.; B. stopped payment at the end of 3 days; in consequence of which the alarm increased, and the daily demand for cash on A. being tripled, A. failed at the end of two more days; but if A. and B. had joined their capitals, they might both have stood the run as it was at first, for 7 days, at the end of which time B. would have been indebted to A. 4000*l*. What was the daily demand for cash on A.'s bank at the beginning of the run?

(59.) What two numbers are those, whose difference, sum, and product are to each other as the three numbers 2, 3, 5?

(60.) A bill of 26*l*. 5*s*. was paid with half-guineas and crowns, and twice the number of half-guineas exceeded three times the number of crowns by 17; how many were there of each?

(61.) One third of a ship belongs to A., and one fifth to B., and A.'s part is worth 1000*l*. more than B.'s; required the value of the ship.

(62.) A person sets out from A., and travels towards B. at the rate of $3\frac{1}{2}$ miles an hour; 40 minutes afterwards another sets out from B. to meet him, travelling at the rate of $4\frac{1}{2}$ miles an hour, and he goes half a mile beyond the middle of the distance before he meets the first traveller; find the distance between A. and B.

(63.) It is required to divide 252 into three parts, such that one third of the first, one fourth of the second, and one fifth of the third, shall all be equal to one another.

(64.) Two labourers, A. and B., received 5*l*. 17*s*. for their wages, A. having been employed 15, and B. 14 days, and A. received for working four days 11*s*. more than B. did for three days; what were their daily wages?

(65.) A person expends half-a-crown in apples and pears, buying his apples at 4, and his pears at 5 a penny; and afterwards accommodates his neighbour with half his apples and one third of his pears, for 13 pence. How many did he buy of each?

(66.) To find a number such that if it be multiplied by 10, and the product be divided by 13, the quotient, increased by the number itself, and by 80, will amount to 1000.

(67.) A person travels a journey at a certain rate; had he travelled half a mile an hour faster he would have performed the journey in $\frac{4}{5}$ of the time, but had he travelled half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road; find the distance, and his rate of travelling.

(68.) Required a number to which if one half of itself, one third of that half, and one fourth of that third, be added, the sum will be 287.

(69.) A person had two casks, the larger of which he filled with ale, and the smaller with cyder. Ale being half-a-crown, and cyder 11*s.* per gallon, he paid 8*l.* 6*s.*; but had he filled the larger with cyder, and the smaller with ale, he would have paid 11*l.* 5*s.* 6*d.*: how many gallons did each hold?

(70.) A father bequeaths to his three sons 7800*l.*, in such a manner that if the share of the eldest be multiplied by 4, that of the second by 6, and that of the third by 8, the products are all equal; what are their shares?

(71.) A person rows from Cambridge to Ely (a distance of 20 miles) and back again in 10 hours, the stream flowing uniformly in the same direction the whole time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it: find the rate of the stream, and the time of his going and returning.

(72.) The number of a gentleman's horses is two fifths of the number of his black cattle, and for every four of the latter he has 11 sheep; required the number of each, the number of the sheep exceeding that of the horses by 141.

(73.) A gamester lost, first, the 6th part, and, secondly, the 10th part of a certain sum of money; he then gained the third part of the same sum: supposing that his gain exceeded his loss by 3*l.*, what was the sum?

(74.) A pedestrian finding that he could walk four times as fast forwards as he could backwards, undertook to walk a certain distance ($\frac{1}{4}$ of it backwards) in a certain time. But the ground being bad, he found that his rate per hour backwards was $\frac{1}{8}$ of a mile less than he had supposed, and that to have won his wager he must have walked forwards 2 miles an hour faster than he did: what is his rate per hour backwards?

(75.) A farm was rated at 3*s.* an acre, and the tenant, on receiving back at his rent-day 10 per cent. of his rent, found that the sum returned amounted to 6*l.* more than the whole rate. The next year the rates were doubled, and he received back 15 per cent. of his rent; but he now found that the sum returned only just paid for the whole rate. What was the rent of the farm, and of how many acres did it consist?

(76.) Two women, at the distance of 150 miles, set out to meet each other—one goes 3 miles in the time the other goes 7; what part of the distance does each travel?

(77.) Two persons, A. and B., can perform a piece of work in 16 days. They work together for 4 days, when A. being called off, B. is left to finish it, which he does in 36 days more: in what time would each do it separately?

(78.) A company of 90 persons consists of men, women, and chil-

dren; the men are 4 in number more than the women, and the children exceed the number of men and women by 10. How many men, women, and children are there in the company?

(79.) The owner of a balloon calculated that if he filled the enclosure which he had hired for the day for 5*l.* with spectators at 2*s.* each, and two persons ascended with him, he should gain $\frac{1}{5}$ of his outlay. The gas and the weather proving bad, he pays but half the price of inflating, and ascends alone from the inclosure a fourth part full, and loses $\frac{1}{6}$ of his outlay. He ascends on the next day with a full balloon, the enclosure $\frac{3}{4}$ filled, and a companion with him. By the whole speculation he gained 10*l.*: what did it cost him to fill his balloon?

(80.) A farm of 864 acres is divided between 3 persons. C. has as many acres as A. and B. together; and the portions of A. and B. are in the proportion of 5 : 11. How many acres has each?

(81.) There is a cistern, into which water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, five-twelfths of the cistern is filled in 4 hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in 10 hours and 40 minutes. In how many hours would each cock fill the cistern?

(82.) A labourer is engaged for 48 days on the following conditions: for every day on which he works he is to receive 2*s.* and his board, but for every day on which he does not work he is to pay 1*s.* for his board. If at the close of his engagement he receives 2*l.* 2*s.*, on how many days must he have worked, and on how many has he been idle?

(83.) A merchant wishing to buy a certain quantity of pimento, the price of which he calculates at the rate of 8*l.* for 5 bags, transmits to his foreign agent the requisite sum of money. Before the order arrives, pimento has risen in value, and the money is sufficient only to buy a quantity less by 18 bags than that which the merchant intended. It appears also that $5\frac{1}{4}$ bags more than $\frac{1}{8}$ of the original quantity will now cost 10*l.* 7*s.* more than they would have done had the price not varied: what was the quantity intended to be purchased?

(84.) A charitable person distributed 5*l.* 14*s.* amongst some poor women and children, giving to each woman 6*s.*, and to each child two; and the number of women was to the number of children as 4 : 7; how many were relieved?

(85.) Some hours after a courier had been sent from A. to B., which are 147 miles distant, a second was sent, who wished to overtake him just as he entered B.; to do which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles is 13 hours and 40 minutes: how many miles does each go per hour?

(86.) The sum of three numbers is 70; and if the second is divided by the first, the quotient is 2, and the remainder 1; but if the third

is divided by the second, the quotient is 3, and the remainder 3: what are the numbers?

(87.) A. and B. start to run a race to a certain post and back again. A. returning meets B. 90 yards from the post, and arrives at the starting place 3 minutes before him. If he had returned immediately to meet B., he would have met him at $\frac{1}{6}$ of the distance between the post and starting place: find the length of the course and the duration of the race.

(88.) A. and B. begin trade, A. with triple the stock of B. They each gain 50*l.*, which makes their stocks in the proportion of 7 to 3. What were their original stocks?

(89.) Two loaded wagons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7, being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then 10 tons. What were the weights at first?

(90.) Two canal boats are despatched from the same place, the first at 6 o'clock in the morning, the other at 4 in the afternoon; the first goes 4 miles an hour, the second 9. How many hours will the second boat take to overtake the first?

(91.) The upper spokes R and r of the hind and fore wheels of a carriage are vertical at starting. After r has made one revolution, its direction is at right angles to the spoke next before R; and when R has made $\frac{3}{8}$ of a revolution, r ascending through its second revolution makes the same angle with the horizontal line through the axle as the spoke next before it. Given that the diameter of the fore wheel : difference of the heights of the axles, as the number of the spokes in the fore wheel : 2; compare the magnitudes of the wheels, and find the number of spokes in each.

(92.) There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, which being increased respectively by 6 and 5, are in the proportion of $\frac{2}{3}$ to $\frac{1}{2}$; required the numbers.

(93.) A gentleman gave away a certain sum in charity to 14 men and 15 women. Had the sum been less by 12*s.*, and only half the number of men relieved, the rest being divided amongst the women, each woman would have received 2*s.* more than each man did; but if there had been only 8 women, and the rest had been divided amongst the men, each man would have received twice as much as each woman. How much money was given away?

(94.) The garrison of a town threatened with siege abandoned the town, and retreated at the rate of 27 miles per day. Two days afterwards a corps was sent in pursuit, with orders to overtake the fugitives in 6 days. How many miles per day must the second party march to accomplish their orders?

(95.) A farmer's rent was 50*l.* a year, and his annual expenditure (including the assessed taxes, which amounted to $\frac{1}{8}$ of his expenses) was such that he was able to pay his landlord only 30*l.* The year following his rent was lowered 20 per cent.; the taxes also were reduced one half, and agricultural produce increased in value $\frac{1}{3}$; in consequence, he was enabled to pay his rent and former debt, and to lay by 5*l.* What was his expenditure and the value of his produce each year?

(96.) A person engaged to reap a field of corn for 5*s.* an acre, but leaving 6 acres not reaped, he received 2*l.* 10*s.* Of how many acres did the field consist?

(97.) When wheat was 5*s.* a bushel, and rye 3*s.*, a man wanted to fill his sack with a mixture of rye and wheat, for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6*s.* left. How must he lay out his money, and fill his sack?

(98.) From the first of two mortars in a battery 36 shells are thrown before the second is ready for firing. Shells are then thrown from both in the proportion of 8 from the first to 7 from the second; the second mortar requiring as much powder for 3 charges as the first does for 4, it is required to determine after how many discharges of the second mortar the quantity of powder consumed by it is equal to the quantity consumed by the first?

(99.) Two persons, A. and B., start at the same time for a race, which lasted 6 minutes. Now after galloping 4 minutes at the same uniform pace at which each started, the distance between them is $\frac{1}{440}$ of the whole length of the course. They continue to run for 1 minute more at the same speed as at first; and then B., who is last, quickens the speed of his horse 20 yards a minute, and comes in exactly 2 yards before A., whose horse has run at the same uniform pace throughout. What is the length of the course?

(100.) A stage coach carries 6 inside, the fare outside is 13*s.*, and one third of the sum of the outside fares exceeds one sixth of those inside by 1*l.* 1*s.* $8\frac{4}{5}$*d.* An opposition arising, the coachman loses three outside and two inside passengers, and also reduces the inside fare by 5*s.*, and halves the outside; and then the whole loss is 7*l.* 0*s.* 6*d.* Find the number of outside places, and the fare inside.

(101.) In a sea fight, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk; 15 escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist?

(102.) A gentleman being asked the age of his two sons, answered, that if to the sum of their ages 18 be added, the result will be double the age of the elder; but if 6 be taken from the difference of their

ages, the remainder will be equal to the age of the younger. What then were their ages?

(103.) A cistern is filled by three pipes, A. B. C.; the pipes A. B. together fill the cistern in 70 minutes; A. C. together in 84 minutes; B. C. together in 140 minutes. In what time will each pipe fill the cistern, and in what time will it be filled if all three pipes are open?

(104.) A. entered into a canal speculation with 14 others, and the profits in this concern amounted in all to 595*l.* more than 5 times the price of an original share. Seven of his former partners joined him in a scheme for navigating the canal with steam-boats; each venturing a sum less than his former gains by 173*l.* But the steam-boats blowing up, A. found he had lost 419*l.* by them; for the company not only never recovered the money advanced, but lost all they had gained by digging the canal, and 368*l.* besides. What were the prices of shares in the two concerns originally?

(105.) To find four numbers such, that the sum of the first, second, and third shall be 13; the sum of the first, second, and fourth, 15; the sum of the first, third, and fourth, 18; and, lastly, the sum of the second, third, and fourth, 20.

(106.) Supposing that 32 lbs. of sea water contain 1 lb. of salt, how much fresh water must be mixed with these 32 lbs. in order that 32 lbs. of the mixture may contain only 2 oz. of salt, or $\frac{1}{8}$ of the former quantity?

(107.) Two clocks are striking the hour together, and are heard to strike 19 times. There is a difference of 2 seconds in their time, and one strikes every 3, the other every 4 seconds. What is the hour they strike; it being observed that, when the clocks strike in the same second, the sounds cannot be distinguished, so as to determine whether one or both strike in that second? and that this is the case with the last stroke of the faster clock?

(108.) To divide 48 into 4 such parts that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3, may be all equal to each other.

(109.) A number is expressed by 3 figures; the sum of the figures is 13; the figure which expresses the simple units of the number is triple the figure which expresses the hundreds; and if 396 is added to the number, the sum is the number reversed. Required the number.

(110.) The gas contractors engage to light a shop with 5 large and 3 small burners; but having by them only one large burner, supply the deficiency with 5 small ones. The shopkeeper, not finding this light sufficient, procures two more small burners, and at the same time agrees for the lights to burn double the usual time on Saturday nights, for which additional gas he was to pay 1*l.* 12*s.* How much did he pay a year altogether?

(111.) A cistern is to be filled with water from three different cocks; from the first it can be filled in 4 hours, from the second in 10, and from the third in 15. How soon would they all together fill it?

(112.) A. and B. are two towns situated on the banks of a river which runs at the rate of 4 miles an hour. A waterman rows from A. to B. and back again, and finds that he is 39 minutes longer on the water than he would have been had there been no stream. The next day he repeats his voyage with another waterman, with whose assistance he can row half as fast again; and they find that they are only 8 minutes longer in performing their voyage than they would have been had there been no stream. Required the rate at which the waterman would row by himself?

(113.) Three brothers, A., B., C., buy a house for 2000*l.*; C. can pay the whole price if B. give him half of his money; B. can pay the whole price if A. give him one third of his money; A. can pay the whole price if C. give him one fourth of his money. How much has each?

(114.) Show at what periods the hands of a watch will be together between 7 and 8 o'clock.

(115.) A farmer sold a certain number of bushels of barley, and ten bushels of wheat, for 7*l.* 19*s.* Now each bushel of wheat cost within 3 shillings as much as two bushels of barley. He afterwards sold as many bushels of barley and four more, and fifteen bushels of wheat, and received two shillings per bushel more for his wheat and barley than he did before; when he found that if he had received 1*l.* 4*s.* more, he should just have received twice as much as he did before. How many bushels of barley did he sell the first time; and what were the prices per bushel of the wheat and barley?

(116.) A farmer laid up a stock of corn, expecting to sell it in six months at three shillings per bushel more than he gave for it. But the price of corn falling one shilling per bushel, he found that by selling it he should lose the price of five bushels. He therefore kept it till the end of the year, and selling it at two shillings per bushel under prime cost, found his loss to be ten shillings less than his expected gain. Required the quantity of corn laid up, and the price per bushel, allowing 5 per cent. simple interest.

(117.) Two men, A. and B., set out from the same place to travel. A. goes in 6 days twice as many miles as B. goes in 5 days, but does not arrive at the end of his journey till 5 days after B. has arrived at the end of his, when he finds that he has travelled 259 miles more than B. But had B. gone 2 miles per day more than he did, and A. stopped 6 days sooner, A. would then have gone only 37 miles more than B. How many miles did each travel per day, and how many days did they travel?

(118.) Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two thirds of the time that Silenus would have taken to empty the whole cask.

After that, Silenus awoke, and drank what Bacchus had left. Had they drunk both together, it would have been emptied two hours sooner, and Bacchus would have drunk only half what he left Silenus. Required the time in which they would empty the cask separately.

(119.) A silversmith has three bars composed of silver, copper, and tin, mixed in different proportions. The pound (avoirdupois) of the first bar contains 7 oz. of silver, 3 oz. of copper, and 6 oz. of tin; the pound of the second contains 12 oz. of silver, 3 oz. of copper, and 1 oz. of tin; and the pound of the third contains 4 oz. of silver, 7 oz. of copper, and 5 oz. of tin. How much of each of these 3 bars must be taken to form a fourth, the pound weight of which shall contain 8 oz. of silver, $3\frac{3}{4}$ oz. of copper, and $4\frac{1}{4}$ oz. of tin?

(120.) Two master bricklayers undertake to lay the foundation of a new court, each taking a certain part, and begin at the same time. If they had continued to work together until the whole was finished, it would have required only $\frac{4}{5}$ of the time it actually took; and in this case B. would do enough to occupy A. 3 months, and A. enough to occupy B. 12 months, which is 36 yards more than A. contracted to do. How many yards did the foundation contain?

(121.) A labourer engages to work for 3*s.* 6*d.* a day and his board, but to allow 9*d.* for his board each day that he is unemployed. At the end of 24 days he has to receive 3*l.* 2*s.* 9*d.* How many days did he work?

(122.) A. and B. playing at billiards, A. bet 5 shillings to 4 on every game, and found that after a certain number of games he had won 10 shillings. Had B. won one game more, the number won by him would have been to the number won by A. as 3 to 4. How many did each win?

(123.) A revenue cutter observes a smuggler q leagues directly to windward; and gives chase, sailing at $5\frac{1}{3}$ points from the wind, and making tacks of $4p$ miles. The smuggler immediately lies off on the other tack at $\frac{8}{3}$ points, making tacks of $\frac{p}{\sqrt{3}}$ miles, its rate of sailing being to the cutter's as $1 : 4\sqrt{3}$. They sail half the above distances before the first tack. In what tack will the smuggler first come within range of the cutter's guns, which carry r miles?

(124.) Three workmen are employed to dig a ditch of 191 yards in length. A. can dig 27 yards in 4 days, B. 35 yards in 6 days, and C. 40 yards in 12 days; in what time could they do it if they worked simultaneously?

(125.) A besieged garrison had such a quantity of bread, as would, if distributed to each at 10 ounces a day, last 6 weeks; but having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces per day for 8 weeks. Required the number of men at first in the garrison.

(126.) A man who is not aware that his watch gains uniformly, engages to ride from Cambridge to London in 9 hours, and sets his watch by St. Mary's at the time of starting. Upon looking at his watch after having gone half way, he supposes it necessary to increase his pace in the ratio of 4 : 3; in consequence of which he arrives in London a quarter of an hour within the time agreed on. But if the watch had lost at the same rate, and he had looked at it at the 14th milestone, and then regulated his pace accordingly, he would have been in London too late by 7 minutes. Find at what rate he set out, and the distance from Cambridge to London by the road he travelled.

(127.) A. and B. travelled on the same road, and at the same rate, from H. to L. At the 50th milestone from L., A. overtook a drove of geese, which were proceeding at the rate of 3 miles in 2 hours, and two hours afterwards met a stage-wagon which was moving at the rate of 9 miles in 4 hours. B. overtook the same drove of geese at the 45th milestone, and met the same stage-wagon exactly 40 minutes before he came to the 31st milestone. Where was B. when A. reached L.?

(128.) A composition of copper and tin containing 100 cubic inches weighed 505 ounces; how many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin to weigh $4\frac{1}{4}$ inches?

(129.) A landlord agrees with his steward to allow him a certain per centage on the rents collected, on condition that he returns half the same per centage on the rents not paid. The first year the steward's income amounts to 6 per cent. on the whole rental; but in the following he finds it necessary, in order to make up the deficiency from his last year's income, to make a return of rents received 270*l.* under their actual value. In the third year, though the rents are reduced $7\frac{1}{2}$ per cent., the amount of rents not paid is the same as in the second year; the steward's income is only $\frac{3}{4}$ of his first year's income, and to make up the deficiency he doubles the amount of his last year's fraud. Required the rental of the estate.

(130.) There are two towns, A. and B., which are 131 miles distant from each other. A coach sets out from A., at 6 o'clock in the morning, and travels at the rate of 4 miles an hour without intermission, in the direct road towards B. At 2 o'clock in the afternoon of the same day a coach sets out from B. to go to A., and goes at the rate of 5 miles an hour constantly. Where will they meet?

(131.) A. lent B. a sum of money to be repaid with interest at the end of a year, and received as security Spanish 5 per cent. bonds to such an amount that their interest was equal to the interest of the debt. At the year's end B. proved insolvent, and Spanish bonds having fallen 40 per cent. A. found that he had lost 400*l.* Had they not fallen in value, he would have been enabled to repay himself, and to return to B. 250*l.*; and had he been at liberty to have sold them

out when they were at 50 (which was before the interest upon them was payable) he would have lost only 300*l*. Required the amount of the debt, and its interest, and the price of Spanish bonds at the beginning of the year.

(132.) A. sets out to ride from Newmarket to London, at the same time that B. and C. leave Hockeril and London to ride to Newmarket. A. meets B. 4 hours before C. overtakes B.; but A. on his return from London, meets C. 1 hour before he meets B., on their way back from Newmarket. It was observed that A. rode 10 miles an hour, and met B. at the same place going and returning. It is required to find the rates of travelling of B. and C., and the distance from London to Newmarket; it being given that Hockeril is equally distant from each.

(133.) A. and B. row between two places, B. in a time during which the minute hand of his watch moves over a certain space: but when the minute hand of A.'s watch has described an equal space, he is obliged to relax his speed, and for the rest of the distance moves only $\frac{2}{7}$ as fast as before. When the stream which flows at a given rate (a) is in their favour, the first part of the distance takes A. 6 times as long as the last, but, when the stream is against him, the two parts are performed in equal times; they are also performed by him in equal times, even if he increase his speed in the ratio of 7 : 5, provided he exchanged watches with B. at starting. Supposing their watches to gain uniformly, find the velocities of A. and B.

XI.

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

(1.) Find two numbers whose difference is 3, and the sum of their squares 89.

(2.) There are two numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers?

(3.) What number is that which added to its square makes 42?

(4.) The sum of the squares of the digits composing a number of two places of figures is 25, and the products of the digits is 12; find the number.

(5.) In a court there are two square grass-plots; a side of one of which is 10 yards longer than the side of the other; and their areas are as 25 to 9. What are the lengths of the sides?

(6.) To find two numbers, such that the less may be to the greater as the greater is to 12, and that the sum of their squares may be 45.

(7.) Find two numbers, such that three times their product is equal to the sum of their squares, and their quotient equal to the difference of their squares.

(8.) A person bought two pieces of linen, which together measured 36 yards. Each of them cost as many shillings per yard, as there were yards in the piece; and their whole prices were in the proportion of 4 to 1. What were the lengths of the pieces?

(9.) What two numbers are those, whose difference is 2, and the difference of their cubes 98?

(10.) The sum of three successive biquadrate numbers is 98; find them.

(11.) There are two numbers, whose sum is to the less as 5 to 2; and whose difference, multiplied by the difference of their squares, is 135. Required the numbers.

(12.) What two numbers are those, whose sum is 6, and the sum of their cubes 72?

(13.) Find two numbers, such that their sum, product, and the difference of their squares may be all equal.

(14.) There are two numbers, which are in the proportion of 3 to 2; the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Required the numbers.

(15.) Find three numbers, such that if the first be multiplied by the sum of the second and third, the second by the sum of the first and third, and the third by the sum of the first and second, the products shall be 26, 50, and 56.

(16.) There is a field in the form of a rectangular parallelogram, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to one-sixth of the whole, being planted, there remain for ploughing 625 square yards. What are the dimensions of the field?

(17.) To divide the number 11 into two such parts, that the product of their squares may be 784.

(18.) The product of two numbers $= a$, their quotient $= b$; required the expressions for the numbers in terms of a, b.

(19.) What three numbers are those which have their differences equal, their sum 15, and the sum of their cubes 495?

(20.) To divide the number 5 into two such parts, that the sum of their alternate quotients may be $4\frac{1}{4}$, that is, of the two quotients of each part divided by the other.

(21.) The sum of the squares of two numbers is 13001, and the difference of their squares is 1449; required the numbers.

(22.) Find three numbers in the proportion of $\frac{1}{2}$, $\frac{2}{3}$. and $\frac{3}{4}$, the sum of whose squares is 724.

(23.) What number added to the numerator and denominator of the fraction $\frac{2}{5}$ in succession, will make the resulting fraction in the former case $2\frac{2}{5}$ times as great as that in the latter ?

(24.) To find two numbers, such that the sum of their squares may be 89, and their sum multiplied by the greater may produce 104.

(25.) A person bought a number of oxen for 112*l.* : if he had had two more for the money, each would have cost him 2*l.* 16*s.* less. Find the number.

(26.) The sum of two numbers is 16, and the quotient of the greater divided by the less is to the quotient of the less by the greater as 25 is to 9. Find them.

(27.) A party at a tavern owed 7*l.* 4*s.*; but in consequence of three of them having no money, each of the rest had to pay 4*s.* more than he otherwise would have done. Required their number.

(28.) What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{3}$, but when 9 is subtracted from it, there remains a number having the same digits inverted ?

(29.) A draper bought a number of pieces of cloth for 33*l.* 15*s.* which he sold at 2*l.* 8*s.* a piece, and gained as much as one piece cost him. How many pieces were there ?

(30.) What two fractions are those whose sum is 1, and the greater divided by the less gives the quotient 10 ?

(31.) A. and B. distribute 1200*l.* each among a number of persons ; A. gives to 40 more than B., and B. gives 5*l.* a piece to each more than A. Find the numbers.

(32.) Divide the number 49 into two such parts, that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$.

(33.) A vintner sold 7 dozen of sherry and 12 dozen of claret for 50*l.*, and he sold 3 dozen more of sherry for 10*l.* than of claret for 6*l.* Find the price of each.

(34.) A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the regiment ; but these being found to be insufficient, each company furnished 3 more men ; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment ?

(35.) A person bought silk for 240*l.* and keeping 10 yards he sells the remainder for 245*l.*, thus gaining 2*s.* a yard upon the prime cost. Find the quantity bought.

(36.) A charitable person distributed a certain sum amongst some

poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received one-third of as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. Now, the men received all together 18*s*. more than the women. How many were there of each?

(37.) A. and B. had 100 eggs, for which they received equal sums; had A. sold as many as B. he would have received 18*d*., and had B. sold as many as A. he would have only got 8*d*. How many had each?

(38.) What number is that which, being added to its square three times, shall make the sum 70?

(39.) A grazier bought a number of sheep for 75*l*., and after losing two he sold the rest for 2*s*. 6*d*. each more than they cost, and gained the prime cost of two sheep. Find the number.

(40.) Required that number which added to its cube shall make the sum 68.

(41.) Two sides of a rectangular plot of ground are as 1 : 3; and if the less side be diminished by 1 yard, and the greater be increased by 28 yards, the plot will be doubled. Find the sides.

(42.) A merchant ventured a certain sum upon a speculation, and found at the end of a year that he had gained 69*l*. This being added to his stock, at the end of another year be found he had gained exactly as much per cent. as in the year preceding. Proceeding in the same manner, each year adding to his stock the gain of the year preceding, he found at the beginning of the fifth year that his stock was to the original stock as 81 to 16. What was the sum he first laid out?

(43.) A person bought a number of 50*l*. railway shares for 900*l*., when they were at a certain discount; and afterwards, when they were at the same premium, sold all but 10 for 550*l*. What number did he buy, and what did he give for each?

(44.) The sum of the squares of two numbers being expressed by a, and the difference of their squares by b, it is required to express the two numbers in terms of a, b.

(45.) The daily receipts on a railway are 1750*l*.: on three of the trains being taken off, the receipts per train increase by 75*l*., the total daily receipts remaining the same as before. How many trains now run?

(46.) A detachment from an army was marching in regular column, with 5 men more in depth than in front: but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detatchment was drawn up in 5 lines. Required the number of men.

(47.) A. sets out to walk to a town 8 miles distant; and 30 minutes

afterwards B. is sent after him, overtakes him, and then returns to the place they started from at the same time that A. reaches the town. If B. walk 4 miles an hour, find A.'s rate.

(48.) A number consisting of two digits, being multiplied by the digit on the left hand, produces 46; but the sum of the digits multiplied by the same digit produces only 10. Required the number.

(49.) If two numbers are to each other as 3 to 4, and the sum of their squares is 324900; what are the numbers?

(50.) A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on, for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

(51.) From two towns, C. and D., being distant 396 miles, two persons, A. and B., setting out at the same time, met each other, after travelling as many days as are equal to the difference of the number of miles they travelled per day, and it appeared that A. had travelled 216 miles. How many miles did each travel per day?

(52.) The sum of the squares of two numbers is b, and the first number is to the second as m is to n. What are the numbers?

(53.) There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the first digit; but if the digits be reversed, and then divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Required the number.

(54.) The difference between the hypothenuse and base of a right-angled triangle is 6, and the difference between the hypothenuse and the perpendicular is 3. What are the sides?

(55.) There are two numbers such that the sum of the products of the first by 4, and of the second by 3, is 53, and the difference of their squares is 15. Required the two numbers.

(56.) In a parcel containing 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each?

(57.) A number consisting of three digits, which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 48; and if 198 be subtracted from it, the digits will be inverted. Required the number.

(58.) A horse-dealer pays a certain sum for a horse, which he afterwards sells for 144*l*., and gains exactly as much per cent. as the horse cost him. How much did the horse cost?

(59.) A farmer received 7*l*. 4*s*. for certain bushels of wheat, and an

equal sum at a price less by 1*s*. 6*d*. per bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there of each?

(60.) During a scarcity, a person wished to make a mixture of 24 bushels, consisting of wheat, oats, and barley, the quantities of each forming an increasing arithmetical progression. Not being able, however, to procure any barley, he mixed additional quantities of wheat and oats, in proportion of 2 to 3, so as to complete his 24 bushels, when he found the whole quantities of wheat and oats to be in proportion of 5 to 7. How many bushels of each did he originally intend to mix?

(61.) To divide 20 into three parts, such that the continual product of all three may be 270, and that the difference of the first and second may be 2 less than the difference of the second and third.

(62.) Two messengers, A. and B., being dispatched at the same time to a place 90 miles distant, A. rode one mile an hour more than B., and arrived at the end of his journey an hour before him. At what rate did each travel per hour?

(63.) The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers?

(64.) To find three numbers in arithmetical progression, such that the sum of their squares may be 56, and the sum arising by adding together thrice the first, and twice the second, and thrice the third, may amount to 32.

(65.) Bought a number of books, consisting of folios, quartos, and octavos, for 96*l*. 12*s*. Fourteen folios (which was the whole number) cost three times as much as all the quartos; and one quarto cost as many shillings as there were quartos. The number of octavos was 32, and their value was such, that 4 of them cost as much as one quarto. Required the value of each, and the number of quartos.

(66.) To find two numbers, such that their product added to their sum may make 47, and their sum taken from the sum of their squares may leave 62.

(67.) To find three numbers having equal differences, and such that the square of the least added to the product of the two greater may make 28, but the square of the greatest added to the product of the two less may make 44.

(68.) The paving of two square court-yards cost 205*l*., a yard of each costing one-fourth of as many shillings as there were yards in a side of the other; and a side of the greater and less together measure 41 yards. Required the length of a side of each.

(69.) Three merchants, A., B., C., on comparing their gains, find that among them all they have gained 1444*l*., and that B.'s gain

added to the square root of A.'s made 920*l*., but if added to the square root of C.'s it made 912. What were their several gains?

(70.) A person buying a number of apples and pears, amounting together to 80, gave twice as much for the apples as pears; but had he bought as many apples has he did pears, and as many pears as he did apples, his apples would have cost 10*d*., and his pears, 3*s*. 9*d*. How many did he buy of each?

(71.) To find three numbers in arithmetical progression, so that the sum of their squares shall be 93; also if the first be multiplied by 3, the second by 4, and the third by 5, the sum of the products may be 66.

(72). The sum of two numbers is 2, and their product is also 2; what are they? Also find two numbers whose sum is a and whose product is b^2.

(73.) A person exchanged a quantity of brandy for a quantity of rum and 11*l*. 5*s*., the brandy and rum being each valued at as many shillings per gallon as there were gallons of that liquor; but had the rum been worth as many shillings per gallon as the brandy was, the whole value of the rum and brandy would have been 56*l*. 5*s*. How many gallons were there of each?

(74.) There are two rectangular vats, the greater of which contains 20 solid feet more than the other; their capacities are in the ratio of 4 to 5, and their bases are squares, a side of each of which is equal to the depth of the other. What are the depths?

(75.) A ship containing 74 sailors, and a certain number of soldiers, besides officers, took a prize. The sailors received each one-third as many pounds as there were soldiers, and the soldiers received 3*l*. a-piece less, and 768*l*. fell to the share of the officers. Had the officers, however, received nothing, the soldiers and sailors might have received half as many pounds per man, as there were soldiers. How many soldiers were there, and how much did each receive?

(76.) A person employed three workmen, whose daily wages were in arithmetical progression. The number of days they worked was equal to the number of shillings that the second received per day. The whole amount of their wages was seven guineas, and the best workman received 28 shillings more than the worst. What were their daily wages?

(77.) The sum of 700*l*. was divided among four persons, whose shares were in geometrical progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

(78.) A poulterer going to market to buy turkeys, met with four flocks. In the second were 6 more than three times the square root of double the number in the first; the third contained three times as many as the first and second; and the fourth contained 6 more than

the square of one-third of the number in the third; and the whole number was 1938. How many were there in each flock?

(79.) There are three numbers in geometrical progression, the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

(80.) There are four numbers in arithmetical progression: the sum of the squares of the first and second is 34, and the sum of the squares of the third and fourth is 130. Required the numbers.

(81.) There are three numbers, the difference of whose differences is 5, their sum is 44, and continued product 1950. What are the numbers?

(82.) There are three numbers in geometrical progression, whose continued product is 64, and the sum of their cubes is 584. Required the numbers?

(83.) A farmer buys m sheep for £p, and sells n of them at 5 per cent. profit. How must he sell the remainder so as to gain 10l. per cent. on the whole?

(84.) There are three numbers in geometrical progression, whose sum is 14, and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

(85.) A. and B. engaged to reap equal quantities of wheat, and A. began half an hour before B. They stopped at 12 o'clock and rested an hour, observing that just half the work was done. B.'s part was finished at 7 o'clock, and A.'s at a quarter before ten. Supposing them to have laboured uniformly, at what time did each commence?

(86.) A butcher bought a certain number of calves and sheep, and for each of the former gave as many shillings as there were sheep, and for each of the latter one-fourth as much. Now had he given 4 shillings more for each of the former, and 2 shillings more for each of the latter, he would have paid 7l. more. But had a sheep cost as much as a calf, he would have expended 56l. 8s. How many did he buy of each, and what were their prices?

(87.) A traveller set out from a certain place, and went 1 mile the first day, 3 the second, 5 the next, and so on, going every day 2 miles more than he had gone the preceding day. After he had been gone three days, a second sets out, and travels 12 miles the first day, 13 the second and so on. In how many days will the second overtake the first?

(88.) There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers?

(89.) A farmer at a fair found the price of an ox equal to that of three sheep, and that he could just dispose of 100l., buying twice as many sheep as oxen. But waiting till the evening, when the price of an ox

fell 1*l.*, and of a sheep 6*s.* 8*d.*, he got for 100*l.* three times as many sheep as oxen, and increased his whole stock by ten more than he would have done in the former case. How many sheep and oxen did he buy, and what was the price of each?

(90.) A person has two pieces of ground, one of which is in the form of an equilateral triangle, and the other of a rectangular parallelogram, one side of which is equal to a side of the triangle, and the other side is 8 yards less. These he plants with trees, at the distance of two yards from each other, and finds that there are 11 more on the rectangle than on the triangle. What are the lengths of the sides?

(91.) There are four numbers in arithmetical progression, whose sum is 28, and their continual product is 585. Required the numbers.

(92.) When 962 men were drawn up it two solid squares, it was found that the front of one contained 18 more men than the front of the other. What was the number of men in each square?

(93.) A gentleman bought a horse for a certain sum, and having re-sold it for 119*l.*, found that he had gained as much per cent. by the transaction as the horse cost him. What was the prime cost of the horse?

(94.) A cask, whose content is 20 gallons, is filled with brandy, a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water; after which the first cask is filled with the mixture; and it appears, that if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off.

(95.) There are two numbers, such that 3 times the sum of their squares multiplied by the less is equal to 26 times the greater, and twice the difference of their squares multiplied by the greater is equal to 15 times the less. Required the two numbers.

(96.) From two towns, which were 168 miles distant, two persons, A. and B., set out to meet each other; A. went 3 miles the first day, 5 the next, 7 the third, and so on; B. went 4 miles the first day, 6 the next, and so on. In how many days did they meet?

(97.) Three persons divide a certain sum of money amongst them in this manner: A. takes the nth part of the whole, and $\frac{a}{n}$£.; B. takes the nth part of the remainder, and $\frac{a}{n}$£., and C. takes the nth part of what then remained, and $\frac{a}{n}$£.; and then nothing was left. Find the sum.

(99.) A mercer sold a piece of cloth for 24*l*., and gained as much per cent. as the cloth cost him. What was the price of the cloth?

(100.) Two detachments of foot being ordered to a station 39 miles distant, the one by marching $\frac{1}{4}$ mile per hour quicker than the other arrives one hour sooner. What was their speed?

(101.) What number is that which, being multiplied by the number of its digits, equals 1012, and if 63 be subtracted from it, its digits will be inverted?

(102.) What two numbers are those whose difference being multiplied by the difference of their squares produces 576, and whose sum multiplied by the sum of their squares is 2336?

(103.) Given the sum of five numbers in arithmetic progression equal to 20, and the sum of their squares 90. What are the numbers?

(104.) The sum of three numbers in a geometrical progression is 91, and the sum of their squares is 4459. What are the numbers?

(105.) The sum of two numbers is 11, and the sum of their third powers 407. Required the numbers.

(106.) A body of men are just sufficient to form a hollow equilateral wedge three deep, and if 597 be taken away, the remainder will form a hollow square four deep, the front of which contains one man more than the square root of the number contained in the side of the wedge. What is the number?

(107.) A vessel which was leaky was furnished with two pumps, which, being worked by A. and B., A. took 3 strokes to 2 of B.'s; but 4 of B.'s threw out as much water as 5 of A.'s. Now, had they pumped together, they would have emptied the vessel in $3\frac{3}{4}$ hours; but B. first worked for as long as it would have taken A. to empty the hold alone, then A. commenced alone, and cleared the hold in $13\frac{1}{3}$ hours from B.'s commencement; but he pumped 100 gallons less than he would have done if they had both pumped together. Required the quantity of water in the hold, and the hourly leakage.

(108) A dealer bought a number of bushels of wheat, expecting to sell it in 6 months, at a profit of 3*s*. per bushel; but prices having fallen 1*s*. per bushel, he found he should lose the price of five bushels. He then kept it 12 months, and sold it at 2*s*. per bushel under prime cost, and found that his loss was 10*s*. less than his expected gain. Now, if interest be allowed at 5 per cent., what was the quantity and price of the wheat bought?

(109.) Out of a vessel containing 24 gallons of brandy there were drawn, at three successive times, a number of gallons, forming an ascending arithmetic series, and the difference of the squares of the extremes was equal to 16 times the mean; the cask having been filled up with water between each draw, it was found that the last draw was only one sixth of its original strength. Required the number of gallons of pure brandy drawn each time.

(110.) P., Q., R., represent three candidates at an election. Q. polled as many plumpers wanting one, as the split votes betwixt P. and R. exceeded those betwixt himself and R.; and the number of split votes betwixt Q. and R. was one more than twice the number betwixt Q. and P. If P. had not voted for himself and R., but for R. only, and if 5 others who split betwixt P. and Q. had voted for Q. only, Q. would have just beaten P., and would have been 48 below R. The number of voters was 1341, of which 565 gave plumpers. Required the number of plumpers for each candidate, and the final state of the poll.

(111.) There is a number consisting of three digits, of which the first is to the second as the second to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it, the digits will be inverted. Required the number.

(112.) Find two numbers whose product is equal to the difference of their squares, and the difference of their cubes equal to the sum of their squares.

(113.) The length of a rectangular field is to its breadth as 6 : 5; one sixth part of the area being planted, there remains 625 square yards. What are the sides of the field?

(114.) Find two numbers such that the product of the greater and the cube of the less may be to the product of the less and the cube of the greater as 4 : 9; and the sum of the cubes of the numbers may be 35.

(115.) A ship with a crew of 175 men set sail with a store of water sufficient to last to the end of the voyage. But in 30 days the scurvy made its appearance, and carried off 3 men every day; and at the same time a storm arose, which protracted the voyage 3 weeks. They were, however, just enabled to arrive in port, without any diminution in each man's daily allowance of water. Required the time of the passage, and the number of men alive when the vessel reached the harbour.

(116.) There are three numbers, the difference of whose differences is 5, their sum is 20, and their continual product 130. Required the numbers.

(117.) Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre; and they altogether can finish the undertaking in 60 days. In how many days can each separately reap an acre?

(118.) The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each.

(119.) A number of persons purchased a field for 345*l.* The youngest contributed a certain sum, the next 5*l.* more, the third 5*l.* more than

the second, and so on to the oldest. For the greater accommodation of the seniors, the field was divided into two parts, the younger half taking a portion proportional to the sum they had subscribed: and in order that each might have an equal share in this portion, they agreed to equalise their contributions, and each to pay 22*l*. Required the number of persons and the sums paid by each.

(120.) Some bees were sitting on a tree; at one time the square root of half their number flew away. Again eight-ninths of the whole flew away the second time; two bees remained. How many were there?

(121.) The number of deaths in a besieged garrison amounted to 6 daily; and, allowing for this diminution, their stock of provisions was sufficient to last for 8 days. But on the evening of the sixth day 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the stock of provisions unconsumed at the end of the sixth day to support 6 men for 61 days; it is required to find how long it would support the garrison, and the number of men alive when the provisions were exhausted.

(122.) D. sets out from F. towards G., and travels 8 miles a day; after he had gone 27 miles, E. sets out from G. towards F., and goes every day $\frac{1}{20}$ of the whole journey; and, after he had travelled as many days as he goes miles in one day, he met D. What is the distance of the two places?

(123.) There are three numbers, the difference of whose differences is 3, their sum is 21, and the sum of the squares of the greatest and least is 137. Required the numbers.

(124.) Three persons A., B., and C., went into a gaming house; the sums which they severally had were in a decreasing geometrical progression. Upon quitting it, they found that the sums which they then had were in a decreasing arithmetical progression; that what B. had remaining was to what he had lost in proportion of the sum to the difference of what he and C. had at first; and that C. had neither won nor lost. If C. had won what A. lost, he would then have had 64*l*. more than A. had remaining; also, the whole sum which they had remaining was to what they had lost as 6 : 7. Required the sums which they had at first.

(125.) A cistern can be filled by 3 different pipes; by the first in $1\frac{1}{3}$ hours, by the second in $3\frac{1}{3}$ hours, and by the third in 5 hours; in what time will this cistern be filled when all three pipes are open at once?

(126.) There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the tens or second digit; but if the digits be inverted, and the resulting number be divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Find the number.

(127.) Find four numbers which exceed one another by unity, such that their continued product may be 120.

(128.) Two boys set off in opposite directions from the right angle of a triangular field, and ran along the sides without varying their velocities, which were in the ratio of 13 : 11. They met in the middle of the opposite side, and afterwards 30 yards from the point where they started. Required the lengths of the sides of the field.

(129.) On January 1st, 1799, a certain beggar received from A. as many groats as A. was years old, who repeated a similar donation every January for the seven following years, during the last of which A. died, his alms to the poor man having in all amounted to 7*l.* 18*s.* 8*d.* Required in what year he was born, and his age at his death.

(130.) A person bought 2 cubical stacks of clover for 41*l.*, each of which cost as many shillings per solid yard as there were yards in a side of the other, and the greater stood on 9 square yards more than the less. What was the price of each?

(131.) A. and B. were travelling along rectilinear roads, which intersected each other. A. had proceeded a yards past the intersection of the roads, when B. arrived at it; after an interval of m minutes, A. was exactly on the left of B.; and n minutes afterwards, B. was on the right of A., and b yards distant from him. Supposing them never to have varied their speed, at what rate did each travel?

(132.) From each of two bags, containing a certain number of balls respectively, a person draws out a handful, and finds that the number remaining in the greater is exactly the cube of that remaining in the less, and exactly the square of one handful. He then draws out of the greater until he finds that the number remaining in it is exactly the square of that remaining in the less; and, also, that if he now emptied the greater into the less, its original number will be increased by two-thirds. What was the number of the balls in each bag at first?

(133.) A farmer laid up a stock of corn, expecting to sell it in 6 months at 3*s.* per bushel more than he gave for it. But the price of corn falling 1*s.* per bushel, he found that by selling it he should lose the cost price of 5 bushels; he therefore kept it to the end of the year, and selling it at 2*s.* per bushel under prime cost, found his loss to be 10*s.* less than his expected gain. Required the quantity of corn laid up, and the price per bushel, allowing 5 per cent. simple interest.

(134.) There are three towns, A., B., and C., the straight lines joining which form a right-angled triangle, B. being situated at the right-angle, and the distance from A. to B. being the least of the three. A pedestrian making a circuit of them, at a uniform rate, finds that the time of his going from A. to B., together with the time of going from B. to C., exceeds the time of C. to A. by 2 hours and

40 minutes. A coach, which left A. 4 hours after the pedestrian, to make the same circuit, overtakes him at the end of the 8th mile from B. to C., the rate of the coach being three times that of the pedestrian; and after reaching A., and waiting there 6 hours and 40 minutes, it starts again to make the same circuit, and arrives again at A., exactly at the same time with the pedestrian, who had rested 4 hours at C. Find the distances of the towns from each other, and the rates of travelling of the pedestrian and coach.

(135.) Three boats A., B., C., start in a race at the same instant; B. being 20 yards behind A., and C. the same distance behind B. A. and B. set off at an uniform rate, C, advancing 1 yard less than A. at every stroke. But B. took 7 strokes to 6 of A.'s or C.'s, and increased its speed besides by 3 inches every stroke; so that by the time A. had taken 42 strokes, B., though it had lost 16 yards by steering, was only 1 yard behind A. At this time B. was observed to fall back, and its velocity decreased twice as fast as it had increased before; whilst C., quickening its strokes at the same instant in the ratio of 6 : 7, and gaining each stroke as much velocity as B. lost, at the end of 28 strokes overtook B., which had lost 11 yards more by steering. Find the velocities with which they started, and the number of yards each made per stroke at first.

(136.) Two persons, A. and B., on comparing the distances they had travelled, found that the square of the number of miles which A. usually walked per hour exceeded the square of the number which B. usually walked by 5; and that if to the square of the product of those numbers there be added the square of the sum of their fourth powers, augmented by the product of the square of the difference of their squares into the square of the product of the numbers themselves, the aggregate amount would be 10345. How many miles did each walk per hour?

(137.) A., B., C., D., are four rough diamonds; the value of C. in pounds is 52 less than the weight of A. in carats, and the value of C. and D. in pounds is equal to the weight of B. in carats; after being cut, each is found to have lost half its weight. The dust from A. and B. is worth 85*l.*; the value of A. is to the value of C. and D., and the dust from A., as the value of B. is to the value of the dust from B. A diamond weighing one carat when rough is worth 3*l.* when cut, and 2*l.* when uncut; the value is proportional to the square of the weight, and a carat of the dust is worth 1*l.* Find the value of D. when cut.

(138.) There are two sorts of metal, each being a mixture of gold and silver, but in different proportions. Two coins from these metals of the same weight are to each other in value as 11 to 17; but if to the same quantities of silver as before in each mixture double the former quantities of gold had been added, the values of two coins from them of equal weights would have been to each other as 7 to 11. Determine the proportion of gold to silver in each mixture, the values of equal weights of gold and silver being as 13 to 1.

(139.) From the towns C. and D. two travellers A. and B., set out to meet each other, A. beginning his journey 3 hours sooner than B. They meet at the distance of 20 miles from D., and A. reaches D. one hour before B. arrives at C. The next day, B. having begun to return at an early hour, meets A., who had then performed only $\frac{1}{4}$ of his journey back; and, notwithstanding a subsequent delay of 3 hours, arrives at D. soon enough, were it necessary, to go 28 miles farther before A. reaches C. Required the distance between the towns, and the rate at which each person travels.

(140.) A person sets off to walk from Cambridge to London, at the rate of $3\frac{1}{2}$ miles an hour. In $2\frac{1}{2}$ hours he is overtaken by the Times, and at 10 minutes before 10 o'clock by the Fly; after resting $2\frac{1}{2}$ hours on the road, he starts again, and meets the Times on its return from London, and half a mile farther, the Fly, at 20 minutes past 5. Supposing the Times and Fly to have started from Cambridge at 6 and half-past 7 o'clock respectively, and from London at 3, determine the distance from Cambridge to London, and the rates at which the coaches travelled.

(141.) A stable-keeper bought 2 horses for 50*l*., and at the end of the year sold one of them for double, and the other for half what he gave for it. The former being well fed and lightly worked, produced for its hire only the half of what it cost him, and consumed in keep as much per cent. on its price as the hire of the other produced on its price, the latter being kept for $\frac{5}{8}$ of as many guineas as it sold for pounds. The keep of the two amounted to 33*l*., and the whole sum that he made by the horses was 9 times his profit on the sale. What did each horse cost?

(142.) From the middle of a town two streets branched off and crossed a river that ran in a straight course, by two bridges, A. and B. From their junction a sewer, equally inclined to both streets led to a point in the river at the distance of 6 chains from the bridge A., and a distance from B. less by 11 chains than the length of the sewer; the expense of making it amounted to as many pounds per chain, as there were chains in the street leading to A. The sewer, however, being insufficient to carry off the water, an additional drain was made from a point in this street distant 4 chains from the bridge A., which entered the river at the same point with the sewer, and was equally inclined to the river and sewer. Now it was found that a drain down the middle of each street, at the rate of 9*l*. per chain, would have cost only 54*l*. more than the expense of the sewer. Required the lengths of the streets and the sewer, and the distance of its mouth from the bridge B.

(143.) A., B., and C., were three architects; A. and B. built 4 warehouses, with flat roofs, each a large one and each a small one; the width of the two large ones being the same, and also that of the two small ones. A. built his as long and as high as they were broad: but B. made the length and height of his small one equal to the

breadth of his large one, in such a manner that the difference between the solid content of those built by A. and those built by B. was 73728 feet. C. also built a warehouse upon a square plot of ground which which was equal to the difference between the ground plots occupied by those which A. built, and found that it would have just stood upon 2688 feet square, if he had added 8 times as many square feet to the ground plot as there were feet in its width. How many feet wide were the several buildings erected by A., B., and C.?

(144.) Into a cubical cistern, eight feet deep, and having an unknown leak, water is poured from two pumps worked by two men, A. and B. They pump together till the vessel is half filled, when B. falls asleep; A. continues pumping till it is three-fourths filled, and then goes away; B. afterwards waking, finds the cistern still half full, and after pumping till it is again three fourths filled departs also, and meeting with A. charges him with leaving his work unfinished. They return together and find the water $1\frac{1}{2}$ inch lower than when B. left. The leak is now discovered and stopped, and by their joint efforts the vessel is filled in half the time in which they had worked together at first. They remark also that $10\frac{1}{8}$ hours had elapsed since they first began pumping, and that B. had worked alone twice as long as A. had. Supposing that a cubic foot contains $15\frac{5}{8}$ gallons, find the quantity of water thrown in by each pump.

(145.) A steamboat sets out from London 3 miles behind a wherry; and when at the same distance a-head of it, overtakes a barge floating down the stream, and reaches Gravesend $1\frac{1}{2}$ hours afterwards. Having waited, in order to land the passengers, $\frac{1}{5}$ of the time of coming down, it starts to return, and meets the wherry in $\frac{3}{4}$ hr., the barge being then $5\frac{1}{4}$ miles a-head of the steamboat; and occupies the same time in returning to London that the wherry did in coming down from thence. Required the distance between London and Gravesend, and the rate of each vessel, the tide being supposed to run out uniformly the whole time.

(146.) A regiment in which there are between 10 and 100 officers and twice as many serjeants, in clearing the streets during a revolution, loses 2 officers; and after storming a barricade, in which 3 more fall and one accidently joins, is obliged to retreat, and loses other 3. Whilst engaged in clearing the streets, the liability of an officer to fall is half that of a serjeant or private; but at the barricades as 4 : 3, and in the retreat as 3 : 4. Also, on leaving their barracks, the number, whose left hand digits are the serjeants and right the officers, is 20 more than 10 times the number of privates; but in coming back (including the officer who joined) it is only 13 more, the number of officers being still greater than 10. Required the state of the regiment at first.

(147.) In the first eruption of the Thames into the Tunnel, the water rose in the vertical shaft 8 times as fast as in the horizontal levels in the second eruption. It was observed also, that if the levels

at the second influx had been 110 feet longer, the velocities of the water ascending in them in the first and second eruptions and when thus increased would have formed an arithmetical progression, the common difference of which was $\frac{1}{6}$ of the difference of the velocities with which the water rose in the shaft in the two eruptions (that in the second being the greater); and had the levels been of the same length at the first as at the second eruption, the whole time of filling would have been half as long again. The tunnel consists of two equal levels, terminated by a vertical shaft of twice the breadth of either of them; the sections of the shaft and levels are supposed to be squares, and the height of the shaft above the upper surface of the levels to equal twice its breadth; the time of filling in the first eruption being 10 minutes less than in the second. Find the time in the second, and the dimensions of the tunnel.

(148.) There are two vessels, P. and Q., containing quantities of fluid in the ratio of 4 : 21, which consist of mixtures of wine and spirits in different proportions: A. pumps a certain quantity out of P. into Q., and then B. pumps out into Q. $\frac{3}{4}$ of what was left; the strength of the mixture Q. is then found to be $\frac{12}{13}$ of its original strength. Now, if when A. stopped, B. had pumped the same quantity as before out of Q. into P., instead of P. into Q., the strength of the mixture P. would have been exactly a mean proportional between the original strengths of P. and Q.; and B. would have pumped out the same quantity of wine that he did before of spirits. Find the proportions of the wine and spirits in each of the vessels at first; and compare the quantities pumped out by A. and B., the strength of spirits being supposed 3 times that of wine.

XII.

INDETERMINATE EQUATIONS AND PROBLEMS.

Find the integral values of the unknown quantities which satisfy the following equations:—

(1.) $7x - 5y = 9$.

$$\text{Here } x = \frac{5y+9}{7} = y + 1 - \frac{2y-2}{7},$$

and $2y-2$ must be a whole number; let then $y=8$,

$$\therefore \frac{2y-2}{7} = \frac{14}{7} = 2, \text{ and } x = 7.$$

If $y = 15$, then $x = 15 + 1 - 4 = 12$;

and the values of y will be 1, 8, 15, 22, &c.

" x " 2, 7, 12, 17, &c.

(2.) $5x - 7y = 21$. (3.) $11x + 35y = 500$.

(4.) $14x - 5y = 7$. (5.) $27x + 16y = 1600$.

(6.) $80x - 17y = 39$. (7.) $3x + 5y = 26$.

(8.) $19x - 117y = 11$. (9.) $11x + 13y = 190$.

(10.) $7x + 13y = 71$. (11.) $13x + 16y = 97$.

(12.) $11x + 7y = 108$. (13.) $49x - 15y = 11$.

(14.) $10x + 9y = 1000$. (15.) $20x - 21y = 38$.

(16.) $19x + 11 = 14y$. (17.) $11x - 18y = 63$.

(18.) $5x + 7y = 29$. (19.) $2x + 3y = 35$.

(20.) $8x + 13y = 159$. (21.) $17x - 49y + 8 = C$.

(22.) $5x + 3y = 78$. (23.) $4x - 5z + 10 = 0$.

(24.) $20x - 21y = 38$, and $3y + 4z = 34$.

(25.) $5x + 4y + z = 272$, and $8x + 9y + 3z = 656$.

(26.) $x + 2y + 3z = 20$, and $4x + 5y + 6z = 47$.

(27.) $2x + 14y - 7z = 341$, and $10x + 4y + 9z = 473$.

(28.) $2x + 5y + 3z = 108$, and $3x - 2y + 7z = 95$.

(29.) $x + 2y + 3z = 20$, and $4x + 5y + 6z = 47$.

(30.) $6x + 7y + 4z = 122$, and $11x + 8y - 6z = 145$

(31.) $20x - 21y = 38$, and $3y + 4z = 34$.

(32.) $7xy - 3y - 5z = 39$. (33.) $3xy - 7x - 7y = 5$

(34.) $10x + 9y + 7z = 58$. (35.) $5xy = 3x + 24$.

(36.) $xy + 2x + 3y = 42$.

(37.) Find a number which, being divided by 39 and 56, leaves remainders 16 and 27 respectively.

(38.) Find the number of solutions of $11x + 15y = 1031$ in positive integers.

(39.) If $x = 3$, $y = 5$, find the possible number of integral solutions of the equations $3x + 4y = 29$, and $7x - 2y = 11$.

(40.) Find two fractions, with 7 and 9 for their denominators, whose sum is $\frac{19}{21}$; and three fractions, having denominators 3, 4, 5, whose sum is $\frac{47}{60}$.

(41.) Find the number of integral solutions of $2x + 3y = 35$.

(42.) Find the number of integral solutions of $20x + 15y + 6z = 171$.

(43.) Find a number which, being divided by 39, leaves a remainder 16, and divided by 56, leaves 27.

(44.) In how many different ways can 20*l.* be paid in half-guineas and half-crowns?

(45.) In how many ways can 140*l.* be paid in guineas and five-shilling pieces only?

(46.) A number of men and women contributed to a charity 50*l.*; each man gave 19*s.*, each woman 11*s.* How many were there of each?

(47.) A person bought sheep and lambs for 8 guineas; the sheep cost 26*s.* each, the lambs 15*s.* How many of each did he buy?

(48.) A farmer selling live stock consisting of oxen and horses, sold each ox for 8*l.*, each horse for 27*l.*, and all the oxen for 97*l.* more than all the horses. How many oxen and horses did he sell?

(49.) Required to pay 50*l.* in guineas and three-shilling pieces only.

(50.) It is required to find a number such that if it is divided by 11 the remainder is 3, and if by 17 the remainder is 10.

(51.) Find the number of solutions that $9x + 13y = 2000$ will admit of in positive integers.

(52.) Find a number which, being divided by 39, leaves a remainder 16, and by 56 a remainder 24.

(53.) Show that the solution of $ax + by = c$ in positive integers is always possible if a be prime to b, and $c > ab - (a + b)$.

(54.) Find two numbers such that their sum shall be equal to the sum of their squares.

(55.) How many fractions are there having 3 and 4 as denominators, whose sum is $3\frac{1}{4}$?

(56.) Find a number which, being divided by 3, 4, 5, successively, leaves the remainders 2, 3, 4.

(57.) Find three square numbers which form an arithmetic progression.

(58.) Find a perfect number, or one which is equal to the sum of all the numbers which divide it without remainder.

(59.) Find the least number which, being divided by 17 and 26, shall leave remainders 7 and 13 respectively.

(60.) Required two square numbers whose difference shall be equal to a given square number (b^2).

(61.) Find the least whole number which, being divided by 28, 19, and 15, shall leave remainders respectively 19, 15, 11.

(62.) Find the numbers which, when divided by 2 and 3, leave the remainders 1 and 2 successively.

(63.) Find the least number which, being divided by 3, 5, 7, and 2, shall leave for remainders 2, 4, 6, and 0, respectively.

(64.) Find two numbers, such that if the first be multiplied by 17, and the second by 26, the first product shall exceed the second by 7.

(65.) In a foundry two kinds of cannon are cast; each of the first sort weighs 16 cwt., and each of the second 25 cwt.; and for the second kind there are used 100 lbs. of metal less than for the first. How many are there of each kind.

(66.) Divide the number 1591 into two parts, such that the one may be divisible by 23 and the other by 34.

(67.) Required three numbers, such that if the first is multiplied by 7, the second by 9, and the third by 11, the first product shall be 1 less than the second, and 2 greater than the third?

(68.) Required three numbers, such that the sum of their products by the numbers 3, 5, 7, respectively, may be 560, and the sum of their products by the numbers 9, 25, 49, respectively, may be 2920.

(69.) Find a number, N, which, being divided by 11, gives the remainder 3; divided by 19, gives the remainder 5; and divided by 29, gives the remainder 10.

(70.) Find two square numbers whose sum is a square.

(71.) Find two square numbers whose difference is a square.

(72.) In how many ways can a person pay a bill of 12*l.* in crowns and guineas?

(73.) In how many ways can 80*l.* be paid in guineas and sovereigns?

(74.) In how many ways can 1000*l.* be paid in crowns and guineas?

(75.) In how many ways can 500*l.* be paid in guineas and 5*l* notes?

(76.) A regiment of foot (less than 1000), when put in column with 13 men in front, wanted 9 men to complete the last rank; with 15 men in front, then 14 were wanting; but with 17 men in front, the ranks were complete. What was the strength of the regiment?

(77.) Is it possible to pay 100*l.* in guineas and moidores (worth 27*s.*) only?

(78.) I owe a person 1*s*, and have nothing about me but sovereigns, and he nothing but dollars, worth 4*s*. 3*d*. each; how must I discharge the debt?

(79.) A man bought 20 birds for 20*s*., consisting at geese of 4*s*., quails at 6*d*., and snipes at 3*d*. each. How many had he of each?

(80.) A jeweller wishes to mix gold of 14, 11, and 9 carats fine, so as to form a composition of 30 oz in weight of 12 carats fine. Find all the ways in which this can be done in whole numbers.

(81.) A person buys 100 head of cattle for 100*l*., viz. oxen at 10*l*. each, cows at 5*l*. each, calves at 2*l*. each, and sheep at 10*s*. each. How many of each did he buy?

(82.) A farmer buys 120 head of cattle, pigs, goats, and sheep, for 400*l*. Each pig costs 4*l*. 10*s*., each goat 3*l*. 5*s*.; and each sheep 1*l*. 5*s*. How many are there of each kind?

(83.) A person wishes to buy 20 animals for 20*l*., sheep at 31*s*., pigs at 11*s*., and rabbits at 1*s*. each. In how many ways can he do so?

(84.) A farmer went to market with 100*l*. to buy geese, sheep, and oxen, at 1*s*. each for the geese, 1*l*. for the sheep, and 5*l*. for the oxen. How many of each did he buy?

(85.) A farmer buys horses and oxen, and gives 31 crowns for each horse, and 20 crowns for each ox, and he finds that the oxen cost him 7 crowns more than the horses. How many of each did he buy?

XIII.

LOGARITHMIC AND EXPONENTIAL EQUATIONS.

Find the value of x and y in the following equations :—

(1.) $12^x = 168$.

$$\therefore\ x \log. 12 = \log. 168,$$

$$\text{and } x = \frac{2\cdot225309}{1\cdot079181} = 2\cdot08.$$

(2.) $10^x = 1250$. (3.) $(15)^x = 1250$, $(16)^x = 1000$.

(4.) $14^x = 63y,\ 17^x = 87y.$ (5.) $2^x \times 3^y = 560,\ 5x = 7y.$

(6.) $20^x = 100.$ (7.) $a^{bx+d} = c.$

(8.) $2^x = 769.$ (9.) $a^{mx} \times b^{nx} = c.$

(10.) $a^{\sqrt{x}} = b.$ (11.) $\text{Log.}\ x = \frac{1}{2}\log.\ a - \frac{1}{4}\log.\ b.$

(12.) $2\log.\ x + 2\log.\ y = 5,\ 2\log.\ x - 2\log.\ y = 1.$

(13.) $8^x \times 9^x = 4{\cdot}9.$

(14.) $\text{Log.}\ x = 2n\log.\ a + 2m\log.\ b - 2p\log.\ c.$

(15.) $2^{x+1} + 4^x = 80.$

(16.) $7^{\frac{x}{2}+\frac{y}{3}} = 2401,\ 6^{\frac{x}{4}+\frac{y}{2}} = 1296.$

(17.) $(7 \times 9^{-\frac{2}{3}})^x = (864)^{\frac{1}{6}}.$

(18.) $\text{Log.}\ x = \log.\ n + \log.\ y,\ ax + by = c.$

(19.) $(a+b)^{2x} \times (a^4 - 2a^2b^2 + b^4)^{x-1} = (a-b)^{2x}.$

(20.) $\text{Log.}_{10}\ x = 3\log._{10}\ a - 2.$

(21.) $bc^{mx} = ab^{nx}.$ (22.) $a^{2x} + 1 = a^{4x}.$

(23.) $a^{2x} - b = a^x.$ (24.) $x^{\frac{y}{x}} = y,\ x^{\frac{p}{q}} = y.$

(25.) $(a^x)^x \times (b^y)^y = c,\ nx = my.$

(26.) $x^{\frac{y}{x}} = y,\ x^3 = y^2.$ (27.) $3^{x+y} = 20 \times 2^x,\ 2x = 5y.$

(28.) $3^{2x} \times 5^{3x-4} = 7^{x-1} \times 11^{2-x}.$

(29.) $3^{x^2-4x+5} = 1200.$ (30.) $x^{x+y} = y^{4a},\ y^{x+y} = x^a$

(31.) $2a^{4x} + a^{2x} = a^{6x}.$

(32.) $(\sqrt{x})^{4\sqrt{x}+4\sqrt{y}} = (\sqrt{y})^{\frac{8}{3}},\ (\sqrt{y})^{4\sqrt{x}+4\sqrt{y}} = (\sqrt{x})^{\frac{8}{3}}.$

(33.) $a^x - a^{-x} = 2c.$ (34.) $x^x - x^{-x} = 3(1 + x^{-x}).$

(35.) $a^x \times b^y = k,\ nx = my.$ (36.) $a^{x^2} \times b^{\frac{1}{2}y^2} = r,\ nx = my.$

(37.) $y^x = x^y,\ x^a = y^b.$

XIV.

RATIO, PROPORTION, AND VARIATION.

(1.) Which is the greater of the two ratios, $15 : 16$, or $16 : 17$; $3 : 5$, or $5 : 8$, and $19 : 20$, or $20 : 21$?

(2.) Compare the ratios $7 : 8$ and $10 : 11$; $19 : 25$ and $56 : 74$.

(3.) When $x : y :: 2 : 1$, which is the greater, $2ax : 3by$ or $3a : 2b$?

(4.) Show that $a : b > ax : bx + y$; but $< ax : bx - y$.

(5.) Prove that $a^3 + b^3 : a^2 + b^2 > a^2 + b^2 : a + b$.

(6.) Which is the greater of the ratios, $a + x : a - x$ and $a^2 + x^2 : a^2 - x^2$, a being $> x$?

(7.) Which is the $>$, $\dfrac{a + x}{a}$ or $\dfrac{4x}{a + x}$; $\dfrac{a^2 - x^2}{a^3 - x^3}$ or $\dfrac{a - x}{a^2 - x^2}$?

(8.) Show that the ratios of $(a \pm x)^3$, $(a \pm x)^4$ are nearly equal respectively to $a \pm 3x$ and $a \pm 4x$; and that of

$$(a \pm x)^{\frac{1}{3}},\ (a \pm x)^{\frac{1}{4}} \text{ to } a \pm \frac{x}{3} \text{ and } a \pm \frac{x}{4}.$$

(9.) Find the ratio compounded of $a : x$, $x : y$, and $y : b$; also of $x + a : x + b$, and $a(x + b) : b(x + a)$.

(10.) Find the ratio compounded of $a+x : a-x$, $a^2+x^2 : (a+x)^2$, and $(a^2-x^2)^2 : a^4-x^4$.

(11.) If $a : b > c : d$, show that $a + c : b + d < a : b$, but $> c : d$.

(12.) If a be the greatest of the four proportionals a, b, c, d; show that $a + d > b + c$; and that $a^n + d^n > b^n + c^n$.

(13.) Prove that $a - x : a + x$ is $>$ or $< a^2 - x^2 : a^2 + x^2$, according as $a : x$ is a ratio of less or greater inequality.

(14.) If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{a + b}{a - b} = \dfrac{c + d}{c - d}$.

(15.) If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, prove that $a : b :: a + c + e : b+d+f$.

(16.) If $\dfrac{a}{b} = \dfrac{c}{d}$, prove that $\dfrac{ma \pm nb}{pa \pm qb} = \dfrac{mc \pm nd}{pc \pm qd}$.

(17.) If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $a : c :: a^2 : b^2$, and $a : d :: a^3 : b^3$.

(18.) Show that if $(a + b)^2 : (a - b)^2 :: b + c : b - c$,

$$\text{then } a : b :: \sqrt{2a - c} : \sqrt{c};$$

$$\text{and if } a : b : c : d,$$

$$\text{then } \sqrt{a - b} : \sqrt{c - d} :: \sqrt{a} - \sqrt{b} : \sqrt{c} - \sqrt{d}.$$

(19.) If $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$, show that $a : b :: c : d$.

(20.) If $x : y :: a^3 : b^3$, and $a : b :: \sqrt[3]{c + x} : \sqrt[3]{d + y}$, show that $cy = dx$.

(21.) Form into proportions the equations

$$ab = a^2 - x^2, \text{ and } x^2 + y^2 = 2ax.$$

(22.) If x be to y in the duplicate ratio of a to b, and a be to b in the sub-duplicate ratio of $a + x$ to $a - y$, then

$$a - y : y :: x + y : x - y.$$

(23.) Find two numbers in the ratio of 3 to 5, such that the difference of their squares : difference of their cubes :: 8 : 147.

(24.) Find two numbers in the ratio of 9 : 16, such that when each number is increased by 15, they shall be in the ratio of 2 to 3.

(25.) Find two numbers which are to each other as 3 : 4, and their sum : sum of their squares :: 7 : 50.

(26.) If $\dfrac{\sqrt[n]{x} - m\sqrt[n]{y}}{\sqrt[n]{x} + m\sqrt[n]{y}} = \dfrac{\sqrt[2n]{x} - m\sqrt[2n]{x - y}}{\sqrt[2n]{x} + m\sqrt[2n]{x - y}}$,

$$\text{then } \frac{x}{y} = \frac{1 \pm \sqrt{5}}{2}.$$

(27.) If x, the first of three magnitudes, x, y, z, $\propto yz$, and $y^2 \propto xz$, then $x \propto z^3$.

(28.) If $s \propto t^2$ when f is constant, and $s \propto f$ when t is constant, also $2s = f$ when $t = 1$; find the equation between f, s, t.

(29.) A passenger in a railway train observes that another train on a parallel line moving in an opposite direction occupies 2″ in passing him; but when in the same direction, it occupies 30″. Compare the speed of the trains.

(30.) If $y \propto x$, and when $x = 3$, $y = 21$; find the equation between x and y.

(31.) If $xy \propto x^2 + y^2$, and when $x = 3$, $y = 4$; find the equation between x and y.

(32.) If $y \propto a^2 - x^2$, and when $x^2 = a^2 - b^2$, $a^2y^2 = b^4$; find the equation between x and y.

(33.) If $y^2 \propto x^2 - a^2$, and when $x = (a^2 + b^2)^{\frac{1}{2}}$, $y = \frac{b^2}{a}$; required the equation between x and y.

(34.) If $y^2 \propto x$, and when $x = a$, $y = \pm 2a$; find the equation between x and y.

(35.) If $ax + by = cx + dy$, show that $x \propto y$.

(36.) If $x \propto y$ and $y \propto z$, show that

$$(ax + by + cz) \propto \left[h\,(xy)^{\frac{1}{2}} + k\,(xz)^{\frac{1}{2}} + l\,(yz)^{\frac{1}{2}}\right].$$

(37.) If $a + b \propto a - b$, show that $a^2 + b^2 \propto ab$; and if $a \propto b$, show that $a^2 - b^2 \propto ab$.

(38.) There are two vessels, A and B, each containing a mixture of water and wine, A in the ratio of 2 : 3, B in the ratio of 3 : 7. What quantity must be taken from each in order to form a third mixture which shall contain 5 gallons of water and 11 of wine?

(39.) If two globes of gold, whose radii are a and a', are melted and formed into one solid globe, what is its radius?

(40.) The value of diamonds $\propto$ as the square of their weights, and the square of the value of rubies $\propto$ as the cube of their weights; a diamond of a carats is worth m times a ruby of b carats, and both together are worth £c. Find the values of a diamond and ruby, each weighing x carats.

XV.

ARITHMETIC, GEOMETRIC, AND HARMONIC PROGRESSION.

In an *Arithmetic Progression*, if a be the first term, d the common difference, n the number of terms, l the last term, and S the sum of n terms; then

$$l = a + (n-1)\,d,\quad S = [2a + (n-1)\,d]\,\frac{n}{2}.$$

Find the sum of the following Arithmetic Series:—

(1.) $2 + 4 + 6 +$ &c. to 16 terms.

(2.) $1 + 3 + 5 +$ &c. to 100 terms and n terms.

(3.) $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} +$ &c. to 7 terms and 14 terms.

(4.) $-5 - 3 - 1 +$ &c. to 8 terms.

(5.) $\frac{2}{3} + \frac{7}{15} + \frac{4}{15} +$ &c. to 6 terms.

(6.) $\dfrac{n-1}{n} + \dfrac{n-2}{n} + \dfrac{n-3}{n} +$ &c. to n terms.

(7.) $6 + \frac{11}{2} + 5 +$ &c. to 25 terms.

(8.) $116 + 108 + 100 +$ &c. to 10 terms.

(9.) $\frac{1}{2} - \frac{2}{3} - \frac{11}{6} -$ &c. to 13 terms.

(10.) $\frac{1}{2} - 1 - \frac{5}{2} -$ &c. to 30 terms.

(11.) $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}$, &c. to 8 terms and 16 terms.

(12.) $\frac{1}{2} + \frac{2}{3} + \frac{5}{6} +$ &c. to 8 terms and 20 terms.

(13.) $\frac{5}{7}, 1\frac{2}{7} +$ &c. to 10 terms and n terms.

(14.) $\dfrac{a-b}{a+b} + \dfrac{3a-2b}{a+b} + \dfrac{5a-3b}{a+b} +$ &c. to n terms.

(15.) $\dfrac{n-1}{n} + \dfrac{n-3}{n} + \dfrac{n-5}{n} +$ &c. to n terms.

(16.) $(a+x)^2 + (a^2+x^2) + (a-x)^2 +$ &c. to n terms.

(17.) $\left(\dfrac{1}{a} - \dfrac{n}{x}\right) + \left(\dfrac{1}{a} - \dfrac{n-1}{x}\right) + \left(\dfrac{1}{a} - \dfrac{n-2}{x}\right) +$ &c. to n terms.

(18.) The sum of an arithmetic series being 1455, the first term 5, and the number of terms 30; find the common difference.

(19.) The sum of an arithmetic series is 91, the common difference 2, and the last term 19. Find the number of terms.

(20.) The first term of an arithmetic series is $3\frac{1}{3}$, the common difference $1\frac{1}{3}$, and the sum 22. Find the number of terms.

(21.) What number of terms of the series 54, 51, 48, &c. must be taken to make 513?

(22.) Required the number of terms, when the first term is 7, the common difference 2, and the sum 40.

(23.) Required the number of terms when the sum $= 2{\cdot}748$, the first term ·034, and the common difference ·0004.

(24.) Find 3 numbers in arithmetic progression whose product $= 120$, and sum 15.

(25.) Write down the second and seventh terms of the arithmetic series, whose fifth and ninth terms are 1 and 9.

(26.) If the sum of three numbers in arithmetic progression be 15, and the sum of the squares 113; what are the numbers?

(27.) Find three numbers in arithmetic progression whose sum is 24, and their product 480.

(28.) The first two terms of an arithmetic series being together $=18$, and the next term being 12, how many terms beginning with the first must be taken to make 78?

(29.) The first term is n^2-n+1, the common difference 2; find the sum of n terms.

(30.) Insert 7 arithmetic means between 1 and $-\frac{1}{3}$.

(31.) Insert four arithmetic means between 2 and -18.

(32.) The fifth and ninth terms of an arithmetic series are 13 and 25; find the seventh term.

(33.) If the sum of three numbers in arithmetic progression be 30, and the sum of their squares 308, what are the numbers?

(34.) If the sum of n arithmetic means between 1 and 19 is to the sum of the first $(n-2)$ of them :: 5 : 3; find the series.

(35.) Divide $\frac{n}{7}(x+4)$ into n parts, so that each term shall exceed the preceding one by a common difference.

(36.) Find four numbers in arithmetic progression, such that the product of the extremes shall be 27, and the product of the means 35.

(37.) If four numbers be in arithmetic progression, whose sum is 24 and product 945, what are the numbers?

(38.) If the nth and mth terms of an arithmetic progression be m and n respectively, find the number of terms whose sum is $\frac{1}{2}\,.\,(m+n)\,.\,(m+n-1)$, and the last term of the series.

(39.) The sum of the first n terms of a series in arithmetic progression is $\left(na-\frac{n+1}{2}\times b\right)\frac{n}{a+b}$. Find the series.

(40.) If a steam engine is observed to pass over 4 feet in the first second, and 88 feet in the sixtieth second of its motion, how far will it travel in the first minute, supposing its motion to be increased each second by a constant quantity?

(41.) The $(n+1)$th term of an arithmetic series is $\frac{ma - nb}{a - b}$. Required the sum of the series to $(2n+1)$ terms.

(42.) Insert 3 arithmetic means between 1 and 11.

(43.) Insert 9 arithmetic means between 1 and -1.

(44.) The sums of n terms of two arithmetical progressions are as $13-7n : 1+3n$; show that their first terms are as $3 : 2$, and their second terms as $-4 : 5$.

(45.) The sum of the first two terms of an arithmetical progression is 4, and the fifth term is 9; find the series.

(46.) The first two terms of an arithmetical progression being together 18, and the next three terms 12; how many terms must be taken to make 28?

(47.) If the latter half of $2n$ terms of an arithmetical series is equal to one-third of the sum of $3n$ terms of the same series, then $n=6$.

(48.) The difference between the sums of m and n terms of an arithmetical progression : the sum of $m+n$ terms :: $m-n : m+n$.

(49.) Determine the relation between a, b, and c, that they may be respectively the pth, qth, and rth terms of an arithmetic series.

(50.) If S, S', S'' be the sums of three arithmetic series, 1 be the first term of each, and the respective differences be 1, 2, 3; prove that $S + S'' = 2S'$.

(51.) If there be p arithmetical progressions, each beginning from unity, whose common differences are 1, 2, 3......p; shew that the sum of their nth terms is

$$= \tfrac{1}{2}.[(n-1).p^2 + (n+1).p].$$

(52.) If a and b are respectively the first term and common difference of an arithmetic series, S_n the sum of n terms, S_{n+1} the sum of $(n+1)$ terms, &c.; prove that

$$S_n + S_{n+1} + S_{n+2} + \text{\&c. to } n \text{ terms} =$$

$$(3n-1).n.\frac{a}{2} + (7n-2).(n-1).n.\frac{b}{6}.$$

(53.) If S_1, S_2, S_3 .... S_p be the sums of p arithmetic progressions continued to n terms, and their first terms be 1, 2, 3, 4, &c., and their common differences 1, 3, 5, 7, &c.; shew that $S_1 + S_2 + S_3 + \ldots + S_p = \tfrac{1}{2}.(np+1)np$.

(54.) In an arithmetic series if the $(p+q)$th term be $= m$, and the $(p-q)$th $= n$, show that the pth term $= \frac{m+n}{2}$, and the qth term $= m - (m-n)\frac{p}{2q}$.

(55.) S_1, S_2, $S_3 \ldots . S_m$ are the sum of m arithmetic progressions to n terms, and their first terms are respectively 1, 3, 5, &c., and their common differences 1, 3, 5, 7; shew that

$$S_1 + S_2 + S_3 \ldots . + S_m = \frac{n^3}{2}(n+1).$$

XVI. In a *Geometric Progression*, if a be the first term, r the common ratio, n the number of terms, and S the sum of n terms; then

$$S = a\frac{r^n - 1}{r - 1}, \text{ and the } n\text{th term} = ar^{n-1};$$

but if r be a proper fraction, and n = infinity, then $S = \frac{a}{1-r}$.

Find the sum of the following series :—

(1.) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}$, &c. to 8 terms.

(2.) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}$, &c. to 6 terms and infinity.

(3.) $4 - 2 + 1 -$ &c. to 12 terms.

(4.) $\frac{1}{3} + \frac{1}{4} + \frac{3}{16} +$ &c. to 10 terms and infinity.

(5.) $3 - 1 + \frac{1}{3} -$ &c. to 8 terms and infinity.

(6.) $\frac{1}{5} + \frac{1}{15} + \frac{1}{45} +$ &c. to 10 terms and infinity.

(7.) $1 - 2x + 2x^2 - 2x^3 +$ &c. to infinity.

(8.) $\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}} + \frac{1}{3}\sqrt{\frac{3}{2}} -$ &c. to 10 terms and infinity.

(9.) $\frac{2}{3} - \sqrt{\frac{2}{3}} + 1 +$ &c. to 8 terms.

(10.) $2 + \sqrt[4]{8} + \sqrt{2} +$ &c. to 12 terms.

(11.) $3 + 9^{\frac{1}{3}} + 3^{\frac{1}{3}} +$ &c. to n terms and infinity.

(12.) $\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{1}{2-\sqrt{2}} + \frac{1}{2} +$ &c. to infinity.

(13.) $1 - \frac{2}{3} + \frac{4}{9} -$ &c. to infinity.

(14.) $1 + 2 - \frac{1}{4} + \frac{1}{32} -$ &c. to infinity.

(15.) $\frac{a}{x}\sqrt{\frac{3}{2}} + \sqrt{\frac{a}{x}} + \frac{2}{3} +$ &c. to n terms.

(16.) $\frac{a}{b} - \frac{a-b}{b^2}x + \frac{a-b}{b^3}x^2 -$ &c. to infinity.

(17.) $\left(\frac{x}{y}\right)^{\frac{1}{2}} - \left(\frac{y}{x}\right)^{\frac{1}{2}} + \left(\frac{y}{x}\right)^{\frac{3}{2}} +$ &c. to infinity.

(18.) $x^p + x^{p+q} + x^{p+2q} +$ &c. to n terms.

(19.) $x - y + \frac{y^2}{x} - \frac{y^3}{x^2} +$ &c. to n terms.

(20.) $\frac{1}{\sqrt{2}(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2})(2+\sqrt{2})} + \frac{1}{(2+\sqrt{2})(3+2\sqrt{2})}$ + &c. to infinity.

(21.) Find 3 geometric means between 2 and 32.

(22.) Find 3 geometric means between $\frac{1}{2}$ and 128, and 9 and $\frac{1}{9}$.

(23.) Find a mean proportional between 2 and 18, and ·05 and ·2.

(24.) Find 3 terms in geometrical progression, whose sum is 14, and the sum of their squares 84.

(25.) Find the nth term and the sum of n terms of the series 1, 5, 13, 29, 61, &c., and 1, 3, 7, 15, 31, &c.

(26.) If the difference of two numbers is 48, and the arithmetic mean exceed the geometric by 18; what are the numbers?

(27.) If the second term of a geometric series be 4 and the fifth 256, find the series.

(28.) If the sum of 3 numbers in a geometric progression be 7, and the sum of their squares 21, what is the series?

(29.) If the product of four numbers in geometric progression be 64 and the sum of their products taken 2 together be 70, what is the series?

(30.) Insert three geometric means between 1 and $\frac{1}{8}$, and between 100 and $\frac{64}{25}$.

(31.) Find a geometric series such that the sum of the first two terms shall be $1\frac{1}{3}$ and of the next two 12.

(32.) In a geometrical progression, if the $(p+q)$th term $= m$, and the $(p-q)$th term $= n$; then will the pth term $= \sqrt{mn}$,

$$\text{and the } q\text{th term} = m\left(\frac{n}{m}\right)^{\frac{p}{2q}}.$$

(33.) If the pth and qth terms of a geometrical progression be P and Q, then will the nth term $= \left(\dfrac{Q^{p-n}}{P^{q-n}}\right)^{\frac{1}{p-q}}$, and the sum of n terms $= \left(\dfrac{Q^{p-n}}{P^{q-1}}\right)^{\frac{1}{p-q}} \left\{\dfrac{P^{\frac{n}{p-q}} - Q^{\frac{n}{p-q}}}{P^{\frac{1}{p-q}} - Q^{\frac{1}{p-q}}}\right\}$.

(34.) If a, b, and c be the pth, qth, and rth terms of a geometrical progression, then will $a^{q-r}\ b^{r-p}\ c^{p-q} = 1$.

(35.) Insert three geometric means between 39 and 3159; also, between 37 and 2997.

(36.) In the geometrical progression $(x-y) + \left(\dfrac{y^2}{x} - \dfrac{y^3}{x^2}\right) + \text{\&c.}$, show that the sum of n terms : sum (n = inf.) :: $x^{2n} - y^{2n} : x^{2n}$.

(37.) Show that the nth term of $\dfrac{1}{\sqrt{2}} + \dfrac{1}{2} + \dfrac{1}{2\sqrt{2}} + \text{\&c.} = \dfrac{1}{\sqrt{2^n}}$, and the sum of n terms $= \dfrac{1}{\sqrt{2^n}}\left(\dfrac{\sqrt{2^n}-1}{\sqrt{2}-1}\right)$.

(38.) Show that $\dfrac{3}{2} - 1 + \dfrac{2}{3} - \text{\&c. to inf.} = \dfrac{9}{10}$.

(39.) Show that $\sqrt{\dfrac{3}{2}} + \sqrt{\dfrac{2}{3}} + \dfrac{2}{3}\sqrt{\dfrac{2}{3}} + \text{\&c.}$, to n terms, $= \sqrt{\dfrac{3}{2}}\left(\dfrac{3^n - 2^n}{3^{n-1}}\right)$; and to inf. $= 3\sqrt{\dfrac{3}{2}}$.

(40.) Show that $\dfrac{1}{3} + \dfrac{1}{6\sqrt{-1}} - \dfrac{1}{12} - \text{\&c. to inf.} = \dfrac{2}{15}(2 - \sqrt{-1})$.

(41.) Show that $\dfrac{a}{(1+x)^n} + \dfrac{ax}{(1+x)^{n+1}} + \dfrac{ax^2}{(1+x)^{n+2}} + \text{\&c. to inf.} = \dfrac{a}{(1+x)^{n-1}}$.

(42.) In a geometrical progression, show that

$$r = \frac{S-a}{S-l}, \quad l(S-l)^{n-1} - a(S-a)^{n-1} = 0,$$

$$r^n - \frac{Sr^{n-1}}{S-l} + \frac{l}{S-l} = 0, \quad S = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}.$$

(43.) Find the geometrical progression, when the sum of the first and second terms is 9, the sum of the first and third is 15; and show how many terms of $6 + 4 + 2\frac{2}{3} +$ &c. amount to 18.

(44.) In any geometrical progression, consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the difference of the odd and even terms.

(45.) In any geometrical progression, the sum of the first and last terms is greater than the sum of any other two terms equidistant from the extremes.

(46.) If S_1, S_2, S_3, &c., S_n, be the sums of n geometrical progressions, whose first terms are a, $2a$, $3a$, &c., na; then will

$$S_1 + S_2 + S_3 + \text{\&c.} + S_n = \frac{n(n+1)}{2}\left(\frac{r^n-1}{r-1}\right)a.$$

(47.) If a, b, c, d, &c., be n quantities in geometrical progression, then will $\frac{1}{a^2-b^2}$, $\frac{1}{b^2-c^2}$, $\frac{1}{c^2-d^2}$, &c. be in geometrical progression, and the sum of n terms will be $\frac{1}{b^{2(n-1)}} \cdot \frac{a^{2n} - b^{2n}}{(a^2-b^2)^2}$.

(48.) If the arithmetic mean between a and b be twice as great as the geometric,

$$\text{then } \frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}.$$

(49.) If a, b, c, d, e, be in *Harmonic Progression*, then

$$a : c :: a-b : b-c;$$
$$b : d :: b-c : c-d;$$
$$c : e :: c-d : d-e;$$

and their reciprocals are in arithmetic progression;

$$\text{i.e. } ab - ac = ac - bc;$$

$$\text{or, } \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

(50.) If a and b be the first two terms of an harmonic series, find the nth term.

(51.) Insert two harmonic means between 6 and 24, and six between 3 and $\frac{6}{23}$.

(52.) The sum of three consecutive terms in harmonic progression is $1\frac{8}{15}$, and the first term is 1; find the series.

(53.) If the arithmetic mean between a and b be equal to m time the harmonic,

$$\text{then } \frac{a}{b} = \frac{\sqrt{m} + \sqrt{m-1}}{\sqrt{m} - \sqrt{m-1}}.$$

and if the geometric be equal to m times the harmonic,

$$\text{then } \frac{a}{b} = \frac{m + \sqrt{m^2-1}}{m - \sqrt{m^2-1}}.$$

(54.) Insert two harmonic means between 2 and 4.

(55.) Insert four harmonic means between 2 and 12.

(56.) Insert n harmonic means between x and y.

(57.) The sum of three terms of a harmonic series is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$; find the series, and continue it both ways.

(58.) The first two terms of a harmonic series are a and b; it is required to continue the series.

(59.) Compare the arithmetic, geometric, and harmonic means between a and b, and show that the geometric mean is a mean proportional between the arithmetic and harmonic.

(60.) The arithmetic mean between two numbers exceeds the geometric by 13, and the geometric exceeds the harmonic by 12; what are the numbers?

(61.) If the geometric mean between two quantities x and y be to the harmonics as $1 : n$, show that

$$x : y :: 1 + \sqrt{1-n^2} : 1 - \sqrt{1-n^2}.$$

(62.) Find the relation which must subsist between a, b, and c, that they may be the pth, qth, and rth terms of an harmonic series.

(63.) The geometric mean between x and y : arithmetic mean :: 4 : 5; find the value of xy^{-1}.

(64.) If S_1, S_2, S_3, &c. denote the sums of an infinite number of infinite decreasing geometric series, whose first terms are a, a^2, a^3, &c. and common ratios r, $2r$, $3r$; prove that

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \text{\&c.} = \frac{a(1-r)-1}{(a-1)^2}.$$

(65.) If S_1, S_2, S_3, &c. S_n be the sums of n geometric series, whose first terms are $a, 2a, 3a \ldots . na$, and r the common ratio in each; prove that

$$S_1 + S_2 + S_3 + \text{\&c. } S_n = \frac{n(n+1)}{2}\left(\frac{r^n-1}{r-1}\right) a.$$

(66.) If S represent the sum of an infinite geometric series, whose first term is a, and common ratio r, S_2 the sum of the squares, S_3 the sum of the cubes &c. of the terms; prove that

$$\frac{1}{S_1} + \frac{1}{S_2} + \text{\&c. to inf.} = \frac{1}{a-1} - \frac{r}{a-r}.$$

XVII.

PERMUTATIONS AND COMBINATIONS.

If n things be taken r together, the

Number of permutations $= n(n-1)(n-2) \ldots . (n-r+1)$.

Number of combinations $= \dfrac{n(n-1)(n-2) \ldots . (n-r+1)}{1 \,.\, 2 \,.\, 3 \ldots . r}$.

If quantities recur, and they be taken altogether,

Number of permutations $= \dfrac{n(n-1)(n-2) \ldots . 3 \,.\, 2 \,.\, 1}{(1 \,.\, 2 \,.\, 3\, p) \times (1 \,.\, 2 \,.\, 3 \ldots . r)\ \text{\&c.}}$.

(1.) If ten letters, a, b, c, &c. be combined 5 together; how many combinations will they make?

(2.) How many different sums may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence?

(3.) How many different sums may be formed with the following coins: a farthing, a penny, a sixpence, a shilling, a crown, a half-sovereign, a guinea, and a moidore?

(4.) At an election where every voter can vote for any number of candidates not greater than the number to be elected, there are 4 candidates and 3 members to be chosen. In how many ways can a man vote?

(5.) How many days can 5 persons be placed in different positions about a table at dinner?

(6.) From a company of 50 men, 4 are draughted off every night to guard; on how many different nights can a different guard be posted, and on how many of these will any particular soldier be engaged?

(7.) Find all the permutations that can be formed out of the letters of the words (1) Baccalaureus, (2) Mississippi, (3) Hippopotamus, (4) Commencement.

(8.) If the number of things : the number of variations, 3 together :: 1 : 20, then the number of things = 6.

(9.) If the number of permutations of n things r together : number $(r-1)$ together :: 10 : 1, and the number of combinations r together : number $(r-1)$ together :: 5 : 3; find n and r.

(10.) If the number of variations of n things 3 together : the number of variations of $(n+2)$ things, 3 together :: 5 : 12; then $n=7$.

(11.) If the number of variations of n things, 4 together : the number of variations of $\frac{2n}{3}$ things, 4 together :: 13 : 2; then $n = 15$.

(12.) If the number of variations of n things, 3 together, be 12 times as great as the number of variations of $\frac{n}{2}$ things, 3 together; what is the number of variations of the same n things n together?

(13.) If the number of combinations of n things, taken 4 together, is to the number taken 2 together :: 15 : 2; then $n = 12$.

(14.) If the number of permutations of $(2n+1)$ things $(n-1)$ together : number of permutations of $(2n-1)$ things n together :: 3 : 5; then $n = 4$.

(15.) If the number of variations of n things, taken 3 together = six times the number of combinations of n things taken 4 together; then $n = 7$.

(16.) If the number of combinations of n things, taken 5 together, is to the number taken 3 together :: 18 : 5; then $n = 12$.

(17.) There is a certain number of things of which, taken 8 together, the variations are 80, and taken 10 together, 960; how many must be taken away from the original number that the combinations of the remaining things, taken 2 together, may be 15?

(18.) The number of permutations of n things taken r together : the number of permutations of n things taken $r-1$ together :: 42 : 1; and the corresponding combinations :: 1 : 1; find the values of n and r.

(19.) How often might a common die be thrown, so as to *expose* five different faces?

(20.) How many combinations can be made in all of 6 things, taken 1, 2, 3, 4, 5, 6 together?

(21.) If the number of combinations of $\frac{n}{3}$ things, 2 together, is 15; $n = 18$.

(22.) If the number of combinations of $\frac{n}{2}$ things, 4 together, is $3\frac{3}{4}$ of the number of combinations of $\frac{n}{3}$ things, 3 together; then $n = 12$.

(23.) If the number of combinations of $n+1$ things, 4 together, is 9 times the number of combinations of n things, 2 together; then $n = 11$.

(24.) If the number of combinations of n things, 3 together, is $\frac{5}{18}$ of the number, 5 together; then $n = 12$.

(25.) How many words of 6 letters may be made out of the 26 letters of the alphabet, with 2 out of the 5 vowels in every word?

(26.) A person wishes to make up as many different dinner parties as he can out of an acquaintance of 24; how many should he invite at a time?

(27.) Show that the number of different combinations of n things, taken 1, 2, 3.... n together, of which p are of one sort, q of another, r of another, &c., $= (p+1)\ (q+1)\ (r+1)....1$.

XVIII.

BINOMIAL THEOREM.

Show that—

(1.) $(a+x)^6 = a^6 + 6a^5x + 15a^4x^2 + 30a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$.

(2.) $(2x-3y)^5 = 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$.

(3.) $(5-4x)^4 = 625 - 2000x + 2400x^2 - 1280x^3 + 256x^4$.

(4.) $\left(\frac{1}{2}x + 2y\right)^7 = \frac{1}{128}x^7 + \frac{7}{32}x^6y + \frac{21}{8}x^5y^2 + \frac{35}{2}x^4y^3 +$
$70x^3y^4 + 168x^2y^5 + 224xy^6 + 128y^7.$

(5.) $(\sqrt{a} \pm \sqrt{b})^4 = a^2 + 6ab + b^2 \pm 4\sqrt{ab}\,(a + b).$

(6.) The middle term of $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^8 = 70a^{\frac{4}{3}}b^{\frac{4}{3}}.$

(7.) The middle term of $(1 + x)^{2n} = \frac{1.3.5....(2n-1)}{1.2.3....n}2^n x^n.$

(8.) The rth term of $(3a - 2x)^{\frac{1}{5}} =$

$$\frac{1.4.9....(5r-11)}{1.2.3....(r-1)} \cdot \left(\frac{2x}{5}\right)^{r-1} \cdot 3a^{\frac{6-5r}{5}}$$

Find the coefficient of—

(9.) x^5 in $(a-x)^9$.

(10.) x^{12} in $(a^5-b^3x^2)^{\frac{5}{2}}$.

(11.) x^9 in $(5a^3-4x^3)^7$.

(12.) x^4 in $(1+3x-x^2)^5$.

(13.) x^r in $(3a + 2x)^{-\frac{1}{3}}$.

Show that—

(14.) $(ax-x^2)^{-\frac{1}{3}} = \frac{1}{(ax)^{\frac{1}{3}}} + \frac{1}{3} \cdot \frac{x^{\frac{2}{3}}}{a^{\frac{4}{3}}} + \frac{2}{9} \cdot \frac{x^{\frac{5}{3}}}{a^{\frac{7}{3}}} + \frac{14}{81} \cdot \frac{x^{\frac{8}{3}}}{a^{\frac{10}{3}}} +$ &c.

(15.) $(a^{\frac{1}{3}} - x^{\frac{1}{3}})^{-6} = \frac{1}{a^2} + \frac{6x^{\frac{1}{3}}}{a^{\frac{7}{3}}} + \frac{21x^{\frac{2}{3}}}{a^{\frac{8}{3}}} + \frac{56x}{a^3} +$ &c.

(16.) $(a^5- x^5)^{-\frac{1}{5}} = \frac{1}{a} + \frac{1}{5} \cdot \frac{x^5}{a^6} + \frac{3}{25} \cdot \frac{x^{10}}{a^{11}} + \frac{11}{125} \cdot \frac{x^{15}}{a^{16}} +$ &c.;

and the $(r+5)$th term =

$$= \frac{1.6.11.16....(16+5r)}{1.2.3.4....(4+r)} \cdot \frac{1}{5^{4+r}} \cdot \frac{x^{20+5r}}{a^{21+5x}}.$$

(17.) $(a^2-x^2)^{-\frac{1}{2}} = a + \frac{1}{2} \cdot \frac{x^2}{a} + \frac{1.3}{2.4} \cdot \frac{x^4}{a^3} + \frac{1.3.5}{2.4.6} \cdot \frac{x^6}{a^5} +$ &c.

(18.) $(a+x\sqrt{-1})^n = a^n + na^{n-1}x\sqrt{-1} - \frac{n.(n-1)}{1.2}a^{n-2}x^2 -$

$$- \frac{n(n-1).(n-2)}{1.2.3}a^{n-3}x^3\sqrt{-1} + \&c.\ldots$$

$$+ (-1)^{\frac{r}{2}} \frac{n(n-1)\ldots.(n-r+1)}{1.2\ldots.r}a^{n-r}x^r + \&c.$$

(19.) $(a+b)^n + (a-b)^n = 2\left[a^n + \frac{n(n-1)}{1.2}a^{-2}b^2 + \right.$

$$+ \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^{n-4}b^4 +$$

$$\left. + \frac{n(n-1)\ldots.(n-5)}{1.2\ldots.6}a^{n-6}b^6 + \ldots.\right].$$

(20.) $(a+b)^n - (a-b)^n = 2\left[na^{n-1}b + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3 + \right.$

$$\left. + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}a^{n-5}b^5 + \&c.\right]$$

(21.) $(a+b)^n = a^n\left[1 + n\frac{b}{a+b} + \frac{n(n+1)}{1.2}\left(\frac{b}{a+b}\right)^2 + \&c.\right]$

(22.) The third term of $(a+b)^{15} = 105a^{13}b^2$.

(23.) The fifth term of $(a^2-b^2)^{12} = 495a^{16}b^8$.

(24.) $(a+b+c)^3 = a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3$.

(25.) $(a+b+c+d)^2 = a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2$.

(26.) $(a+2b-c)^3 = a^3 + 6a^2b + 12ab^2 + 8b^3 - 3a^2c - 12abc - 12b^2c + 3ac^2 + 6bc^2 - c^3$.

(27.) $\left(\frac{1+2x}{1+x}\right)^n = 1 + n.\frac{x}{1+2x} + n.\frac{n+1}{2}\left(\frac{x}{1+2x}\right)^2 +$

$$+ \frac{n(n+1)(n+2)}{1.2.3}\left(\frac{x}{1+2x}\right)^3 + \&c.\ldots$$

$$+ \frac{n(n+1)\ldots.(n+r-2)}{1.2\ldots.(r-1)}\left(\frac{x}{1+2x}\right)^{r-1} + \&c.$$

(28.) $\left(x+\dfrac{1}{x}\right)^n = x^n + x^{-n} + n\left(x^{n-2}+x^{-(n-2)}\right) + n\,.\,\dfrac{n-1}{2} \times$

$$\left(x^{n-4}+x^{-(n-4)}\right) + \&c.\ldots.$$

$$+\frac{n\,(n-1)\ldots.\left(\dfrac{n}{2}+1\right)}{1\,.\,2\ldots.\dfrac{n}{2}},\ n \text{ even,}$$

$$+\left(x+x^{-1}\right)\frac{n\,(n-1)\ldots.\dfrac{n+3}{2}}{1\,.\,2\ldots.\dfrac{n-1}{2}},\ n \text{ odd.}$$

(29.) $\left(\dfrac{1+x}{1-x}\right)^n = 1 + n\,\dfrac{2x}{1+x} + \dfrac{n\,(n+1)}{1\,.\,2}\,.\left(\dfrac{2x}{1+x}\right)^2 +$

$$\frac{n\,(n+1)\,(n+2)}{1\,.\,2\,.\,3}\,.\left(\frac{2x}{1+x}\right)^3 + \&c.$$

(30.) $\dfrac{1}{\sqrt{1-x^2}} = 1 + \dfrac{1}{2}x^2 + \dfrac{3}{8}x^4 + \dfrac{5}{16}x^6 + \&c.\ldots. +$

$$+\frac{1\,.\,3\,.\,5\,..\,(2r-3)}{2^{r-1}\,.\,1\,.\,2\,.\,3\,..\,(r-1)}\,x^{2(r-1)} + \&c.$$

(31.) $(1-x^2)^{\frac{7}{3}} = 1 - \dfrac{7}{3}x^2 + \dfrac{14}{9}x^4 - \dfrac{14}{81}x^6 - \&c.\ldots.$

$$-\frac{7\,.\,4\,.\,1\,.\,2\,.\,5\ldots.(3r-13)}{3^{r-1}\,.\,1\,.\,2\ldots.(r-1)}\,x^{2(r-1)} + \&c.$$

(32.) $\dfrac{1}{\sqrt[3]{1-x}} = 1 + \dfrac{1}{3}x + \dfrac{2}{9}x^2 + \dfrac{14}{81}x^3 + \&c.\ldots.$

$$+\frac{1\,.\,4\,.\,7\ldots.(3r-5)}{3^{r-1}\,.\,1\,.\,2\ldots.(r-1)}\,x^{r-1} + \&c.$$

(33.) $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}} = 1 + \dfrac{x}{a+x} + \dfrac{3}{2}\cdot\dfrac{x^2}{(a+x)^2} + \dfrac{5}{2}\cdot\dfrac{x^3}{(a+x)^3} + \&c.\,.\,.$

$$+\frac{1\,.\,3\,.\,5\ldots.(2r-3)}{2\,.\,4\,.\,6\ldots.2(r-1)}\cdot\frac{x^{r-1}}{(a+x)^{r-1}} + \&c.$$

(34.) $\frac{1}{(a^2+x^2)^2} = a^{-4} - 2a^{-6}\,x^2 + 3a^{-8}\,x^4 - 4a^{-10}\,x^6 + \&c.\ldots.. + (-1)^{r-1}\,.\,ra^{-2(r+1)}x^{2(r-1)} + \&c.$

(35.) Show that the approximate values of $\sqrt[5]{39}$, $\sqrt[3]{65}$, $\sqrt[4]{260}$, $\sqrt[7]{108}$, $= 2{\cdot}087$, $4{\cdot}02083$, $4{\cdot}01563$, $1{\cdot}95504$ respectively.

(36.) Show that the coefficients of x^n in the expansion of $(1-2x)^2 + (1-x)^4 = \frac{1}{6}(n-6)\ (n^2-1)$.

(37.) Given that the coefficient of the $(p+1)$th term in the expansion of $(1+x)^n =$ that of the $(p+3)$th term; show that $2p = n-2$.

(38.) Find the sum of the coefficients of the expansions of $(a+x)^n$ and $(a-x)^n$.

(39.) Show that the coefficients of x^3 in the expansion of $(a+bx+cx^2)^{\frac{3}{2}}$
$$= \frac{b}{4a^{\frac{1}{2}}}\left(3c - \frac{b^2}{4a}\right).$$

(40.) Show that the coefficients of x^r in the expansion of
$$(1+2x+3x^2\ldots.)^2 = \tfrac{1}{6}(r+1)\ (r+2)\ (r+3).$$

(41.) Show that the $(n+1)$th term of the series whose $(r+1)$th term
$$= \frac{n\,.\,(n-1)\ldots.(n-r+1)}{1\,.\,2\ldots.r}\left(\frac{x}{1+2x}\right)^r = \left(\frac{x}{1+2x}\right)^n.$$

(42.) If the coefficients of x in the 5th and 7th terms of $(1+2x)^n$ are 1120 and 1792 respectively; show that $n = 8$.

(43.) If the coefficients of x in the 3rd and 5th terms of $(1-x)^n$ are $\frac{14}{9}$ and $-\frac{7}{243}$ respectively; show that $n = \frac{7}{3}$.

(44.) If generally n_r be the coefficient of the $(r+1)$th term of $(1+x)^n$, show that $(n+p)_r = n_r + n_{r-1}\,.\,p_1 + n_{r-2}\,.\,p_2 + \&c. + n_1\,p_{r-1} + p_r$.

(45.) If generally m_r be the coefficient of the $(r+1)$th term of $(1-x)^{-m}$, show that $m_r + (m+1)_{r-1} = (m+1)_r$.

(46.) Find the sum of the squares of the coefficients in the expansion of $(1+x)^n$, when n is a positive integer.

(47.) In the expansion of $(1+x)^m$, if P, Q be any two consecutive coefficients, prove that the coefficients after Q will be

$$Q \cdot \frac{Qm - P}{P(m+2) + Q}, \quad Q \cdot \frac{Qm - P}{P(m+2) + Q} \cdot \frac{Q(m-1) - 2P}{P(m+3) + 2Q}, \text{ \&c.}$$

(48.) If R be the rth term of the expansion of $(1-x)^m$, prove that the series after the first r terms will be represented by

$$Rx\left(1 - \frac{m+1}{r}\right) + Rx^2\left(1 - \frac{m+1}{r}\right)\left(1 - \frac{m+1}{r+1}\right) +$$

$$Rx^3\left(1 - \frac{m+1}{r}\right)\left(1 - \frac{m+1}{r+1}\right)\left(1 - \frac{m+1}{r+2}\right) + \text{ \&c.}$$

(49.) Given P and Q the pth and qth terms of the expansion of $(a+x)^m$, find m.

XIX.

INDETERMINATE COEFFICIENTS.

Show by indeterminate coefficients that—

(1.) $\dfrac{x}{x^2 + 6x + 8} = \dfrac{2}{x+4} - \dfrac{1}{x+2}$.

(2.) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \text{ \&c.}$

(3.) $\dfrac{x+c}{(x-a)\,(x-b)} = \dfrac{a+c}{(a-b)\,(x-a)} - \dfrac{b+c}{(a-b)\,(x-b)}$.

(4.) $\dfrac{2x^2-6x+6}{(x-1)\,(x-2)\,(x-3)} = \dfrac{1}{x-1} - \dfrac{2}{x-2} + \dfrac{3}{x-3}$.

(5.) $\dfrac{1-x-4x^2}{x-4x^2+3x^3} = \dfrac{1}{x} + \dfrac{2}{1-x} + \dfrac{1}{1-3x}$.

(6.) $\dfrac{x^2}{(x+1)\,(x+2)\,(x+3)} = \dfrac{1}{2(x+1)} - \dfrac{4}{x+2} + \dfrac{9}{2(x+3)}$.

(7.) $\dfrac{x+2}{x^3-x} = \dfrac{1}{2(x+1)} + \dfrac{3}{2(x-1)} - \dfrac{2}{x}.$

(8.) $\dfrac{13+21x+2x^2}{1-5x^2+4x^4} = \dfrac{1}{1+x} - \dfrac{6}{1-x} + \dfrac{2}{1+2x} + \dfrac{16}{1-2x}.$

(9.) $\dfrac{x^3+x^2+2}{x(x+1)^2(x-1)^2} = \dfrac{2}{x} - \dfrac{1}{2(x+1)^2} - \dfrac{5}{4(x+1)} + \dfrac{1}{(x-1)^2}$
$- \dfrac{3}{4(x-1)}.$

(10.) $\dfrac{3x^2-1}{(x-1)^2(x^2+1)} = \dfrac{1}{(x-1)^2} + \dfrac{2}{x-1} - \dfrac{2x}{x^2+1}.$

(11.) $\dfrac{1}{a^4-x^4} = \dfrac{1}{4a^3(a+x)} + \dfrac{1}{4a^3(a-x)} + \dfrac{1}{2a^2(a^2+x^2)}.$

(12.) $\dfrac{1}{x^4-1} = \dfrac{1}{4(x-1)} - \dfrac{1}{4(x+1)} - \dfrac{1}{2(x^2+1)}.$

(13.) $\dfrac{x+3}{x^4-1} = \dfrac{1}{x-1} - \dfrac{1}{2(x+1)} - \dfrac{x+3}{2(x^2+1)}.$

(14.) $\dfrac{7x+8}{(x^2+1)(x+1)^2} = \dfrac{1}{2(x+1)^2} + \dfrac{4}{x+1} - \dfrac{8x-7}{2(x^2+1)}.$

(15.) $\dfrac{x^2}{(a^2+x^2)(b^2+x^2)} = \dfrac{a^2}{(a^2-b^2)(a^2+x^2)} - \dfrac{b^2}{(a^2-b^2)(b^2+x^2)}.$

(16.) $\dfrac{x^2+x+1}{x^3(x^2+1)^2} = \dfrac{1}{x^3} + \dfrac{1}{x^2} - \dfrac{1}{x} - \dfrac{x-1}{(x^2+1)^2} + \dfrac{2x-1}{x^2+1}.$

(17.) $\dfrac{2x^5-x^2}{(x^2+1)^2(x^2+x+1)} = \dfrac{4}{x^2+1} - \dfrac{2x+1}{(x^2+1)^2} + \dfrac{1}{(x^2+x+1)^2} -$
$\dfrac{4}{x^2+x+1}.$

(18.) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \text{\&c.}$

(19.) $\dfrac{1+2x}{1-x-x^2} = 1 + 3x + 4x^2 + 7x^3 + \text{\&c.}$

(20.) $\dfrac{x^2}{x^2+2ax+a^2} = 1 - \dfrac{2a}{x} + \dfrac{3a^2}{x^2} - \dfrac{4a^3}{x^3} + \text{\&c.}$

(21.) $\dfrac{a-bx}{a+cx} = 1 - (b+c)\dfrac{x}{a} + (b+c)\dfrac{cx^2}{a^2} - (b+c)\dfrac{c^2x^3}{a^3} + \text{\&c.}$

(22.) $\dfrac{x-bx^3+dx^5}{1-ax^2+cx^4} = x + (a-b)x^3 + (a^2-ab-c+d)x^5 + (a^3-a^2b$ $-2ac+ad+bc)x^7 + \text{\&c.}$

(23.) If $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \text{\&c.}$, shew that $y = x + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \dfrac{x^4}{1.2.3.4} + \text{\&c.}$

(24.) If $x = y - y^2 + y^3 - \text{\&c.}$, shew that $y = x + x^2 + x^3 + x^4 + \text{\&c.}$

(25.) If $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{4}$ &c., shew that $y = x + \dfrac{x^2}{2} + \dfrac{x^3}{4} + \dfrac{x^4}{8} + \text{\&c.}$

(26.) If $x = y + y^3 + y^5 + \text{\&c.}$, shew that $y = x - x^3 + 2x^5 - 5x^7 + \text{\&c.}$

(27.) If $x = y - \dfrac{y^3}{3} + \dfrac{y^5}{5} - \text{\&c.}$, shew that $y = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{351}x^7 + \text{\&c.}$

(28.) If $y + y^2 + y^3 + y^4 + \text{\&c.} = x + \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{8}x^4 + \text{\&c.}$, shew that $y = x - \frac{1}{2}x^2 + \frac{1}{4}x^3 - \frac{1}{8}x^4 + \text{\&c.}$

(29.) If $y^3 - axy - b^3 = 0$, shew that $y = b + \dfrac{ax}{3b} - \dfrac{a^3x^3}{3^4b^5} + \dfrac{a^4x^4}{3^5b^7} - \text{\&c.}$

(30.) If $my^3 - xy = m$, shew that $y = 1 + \dfrac{x}{3m} - \dfrac{x^3}{3^4m^3} + \dfrac{x^4}{3^5m^4} - \text{\&c.}$

(31.) If $x = y - \dfrac{1}{1.2.3}y^3 + \dfrac{1}{1.2.3.4.5}y^5 - \text{\&c.}$, shew that $y = x + \dfrac{1}{1.2.3}x^3 + \dfrac{3}{1.2.4.5}x^5 + \dfrac{3.5}{1.2.4.6.7}x^7 + \text{\&c.}$

(32.) If $x = y + \frac{y^2}{a} + \frac{2}{3}\frac{y^3}{a^2} + \frac{2}{3}\frac{y^5}{a^4} + \frac{32}{45}\frac{y^6}{a^5}$ + &c., shew that

$$y = x - \frac{x^2}{a} + \frac{4x^3}{3a^2} - \frac{7x^4}{3a^3} + \frac{14x^5}{3a^4} - \text{\&c.}$$

(33.) If $x = y + \frac{1}{6}y^3 + \frac{1}{24}y^5 + \frac{61}{5040}y^7$ + &c., shew that $y = x$

$$- \frac{1}{6}x^3 + \frac{1}{24}x^5 - \frac{61}{5040}x^7 + \text{\&c.}$$

Find by the method of indeterminate coefficients the sum of—

(34.) $1^2 + 2^2 + 3^2 + 4^2$ + &c. to 11 terms.

(35.) $1^2 + 4^2 + 7^2 + 10^2$ + &c. to n terms.

(36.) $1 + 2x + 3x^2 + 4x^3$ + &c. to n terms.

(37.) $1.2 + 2.3 + 3.4$ + &c. to 10 terms.

(38.) $1.2^2 + 2.3^2 + 3.4^2$ + &c. to n terms.

(39.) $1 + 2^2x + 3^2x^2 + 4^2x^3$ + &c. to n terms.

(40.) $1^3 + 2^3 + 3^3$ + &c. to 20 terms.

Find the sum to n terms of each of the following series:

(41.) $1.2+3.4+5.6$+&c.

(42.) $1+4x+7x^2+10x^3$+&c.

(43.) $1^2+2^2x+3^2x^2+4^2x^3$+&c.

(44.) $1^2+3^2x+5^2x^2+7^2x^3$+&c.

(45.) $1.2+2.3+3.4+4.5$+&c.

(46.) $1.2.3+2.3.4+3.4.5$+&c.

(47.) $1.2.5+3.4.7+5.6.9$+&c.

(48.) $2.5.6+4.7.8+6.9.10$+&c.

(49.) $1^3+2^3+3^3+4^3$+&c.

(50.) $1.1^3+3.2^3+5.3^3+7.4^3$+&c.

(51.) The sum of 3 + 11 + 31 + 69 + 131 + &c. to 20 terms.

(52.) The sum of 1 + 11 + 19 + 30 + 48 + 76 + &c. to 20 terms.

(53.) The sum of 1+6+21+56+126+252+462+&c. to 15 terms.

XX.

VANISHING FRACTIONS.

(1.) $\frac{x^3-1}{x^3-4x^2+4x-1}$ (when $x=1$) $= -3$.

(2.) $\dfrac{x^2+2x-35}{x^2-6x+5}$ (when $x=5$) $=3$.

(3.) $\dfrac{1-3x^2+2x^3}{(1-x)^2}$ (when $x=1$) $=3$.

(4.) $\dfrac{x^3-2x^2-x+2}{x^3-7x+6}$ (when $x=2$) $=\dfrac{3}{5}$.

(5.) $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}\left(\text{if } x=\dfrac{4ab}{a+b}\right)=2$.

(6.) $\dfrac{a^n}{2na^n-2nx}+\dfrac{b^n}{2nb^n-2nx}\left(\text{if } x=\dfrac{a^n+b^n}{2}\right)=\dfrac{1}{n}$.

(7.) $\left(\dfrac{x^3-a^3}{x^2-a^2}\right)^2$, (when $x=a$) $=\dfrac{9a^2}{4}$.

(8.) $\dfrac{x^3-2ax^2-a^2x+2a^3}{x^3-13a^2x+12a^3}$ (when $x=a$) $=\dfrac{1}{5}$

(9.) $\dfrac{x^4-81}{x-3}$ (when $x=3$) $=108$.

(10.) $\dfrac{x^3-1}{x^3-2x^2+2x-1}$ (when $x=1$) $=3$.

(11.) $\dfrac{1+x-x^2-x^3}{1+2x+2x^2+2x^3+x^4}$ (when $x=-1$) $=1$.

(12.) $\dfrac{x^3-5x^2+8x-6}{x^3-7x^2+16x-12}$ (when $x=3$) $=5$.

(13.) $\dfrac{x\,.\,\epsilon^{2x}+2-2\epsilon^{2x}-x}{\epsilon^x-1}$ (when $x=0$) $=-4$.

(14.) $\dfrac{2}{x^2-1}-\dfrac{1}{x-1}$ (when $x=1$) $=-\dfrac{1}{2}$.

(15.) $\dfrac{a(x^2+c^2)-2acx}{b(x^2+c^2)-2bcx}$ (when $x=c$) $=\dfrac{a}{b}$.

(16.) $\dfrac{(x+a)^{\frac{m}{n}}-x^{\frac{m}{n}}}{a}$ (when $a=0$) $=\dfrac{m}{n}x^{\frac{m-n}{n}}$.

(17.) $\dfrac{x^2+y^2+2xy-4}{x+y-2}$ (when $x=1$, $y=1$) $=$

(18.) $\dfrac{x^5 + ax^4 - a^4x - a^5}{x^4 + 2ax^3 + 2a^2x^2 + 2a^3x + a^4}$ (when $x = -a$) $= -2a$.

(19.) $\dfrac{x^4 + 3x^3 - 7x^2 - 27x - 18}{x^4 - 3x^3 - 7x^2 + 27x - 18}$ (when $x = 3$, and when $x = -3$) $= 10$ and $\frac{1}{10}$.

(20.) $\dfrac{x^4 - 4x^3 + 8x^2 - 16x + 16}{x^4 - 6x^3 + 13x^2 - 12x + 4}$ (when $x = 2$) $= 8$.

(21.) $\dfrac{x^3 - 19x + 30}{x^3 - 2x^2 - 9x + 18}$ (when $x = 2$, and $x = 3$) $= \frac{7}{5}$ and $\frac{4}{3}$.

(22.) $\left(\dfrac{1}{x^2 - x} + \dfrac{1}{4x^2}\right)^{\frac{1}{2}} - \dfrac{1}{2x}$ (when $x = 0$) $= -1$.

(23.) $\dfrac{\sqrt{x + 1} - \sqrt{2}}{x - 1}$ (when $x = 1$) $= \dfrac{1}{2\sqrt{2}}$.

(24.) $\dfrac{\sqrt{3x + 1} - 2}{x - 1}$ (when $x = 1$) $= \frac{3}{4}$.

(25.) $\dfrac{1 - ax}{1 + ax}\sqrt{\dfrac{1 + bx}{1 - bx}}$ (when $x = \dfrac{1}{a}\sqrt{\dfrac{2a}{b} - 1}$) $= 1$.

(26.) $\dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}$ $\left(\text{when } x = \dfrac{2ab}{b^2 + 1}\right) = b$.

(27.) $\dfrac{a - (a^n - x^n)^{\frac{1}{n}}}{x^n}$ (when $x = 0$) $= \dfrac{1}{na^{n-1}}$.

(28.) $\dfrac{\sqrt{a + x} - \sqrt{2a}}{\sqrt{a + 2x} - \sqrt{3a}}$ (when $x = a$) $= \dfrac{\sqrt{6}}{4}$.

(29.) $\dfrac{(x^2 - a^2)^{\frac{3}{2}} + (x - a)}{(1 + x - a)^3 - 1}$ (when $x = a$) $= \dfrac{1}{3}$.

(30.) $\dfrac{a - \sqrt{2ax - x^2}}{a - (2a^2x - ax^2)^{\frac{1}{3}}}$ (when $x = a$) $= \dfrac{3}{2}$.

(31.) $\dfrac{\sqrt{2a^2 + 2x^2} - 2(a^2x)^{\frac{1}{3}}}{x - a}$ (when $x = a$) $= \dfrac{1}{3}$.

(32.) $\dfrac{\sqrt{2a^3x - x^4} - a\,(a^2x)^{\frac{1}{3}}}{a - (ax^3)^{\frac{1}{4}}}$ (when $x = a$) $= \dfrac{16a}{9}$.

(33.) $\dfrac{\sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}}$ (when $x = 0$) $= \sqrt{a}$.

(34.) $\dfrac{(x - c)\sqrt{x + c} + \sqrt{x - c}}{\sqrt{2c} - \sqrt{x + c} + \sqrt{x - c}}$ (when $x = c$) $= 1$.

(35.) $\dfrac{a\,(4a^3 + 4x^3)^{\frac{1}{3}} - ax - a^2}{\sqrt{2a^2 + 2x^2} - a - x}$ (when $x = a$) $= 2a$.

(36.) $\dfrac{x - a + \sqrt{2ax - 2a^2}}{\sqrt{x^2 - a^2}}$ (when $x = a$) $= 1$.

(37.) $\dfrac{x\sqrt{3a^3x - 2x^4} - ax\,(a^4x)^{\frac{1}{5}}}{a - (ax^2)^{\frac{1}{3}}}$ (when $x = a$) $= \dfrac{81a^2}{20}$.

(38.) $\dfrac{\sqrt{x^2 + a} - \sqrt{a^2 + x} + \sqrt{a^2 - x^2}}{\sqrt{a^3 - x^3}}$ (when $x = a$) $= \sqrt{\dfrac{2}{3a}}$.

(39.) $\dfrac{(a^2x + 7a^3)^{\frac{1}{3}} - \sqrt{2ax + 2a^2}}{(a^2x^6 + 15a^5x^3)^{\frac{1}{4}} - (15a^4x^6 + 17ax^9)^{\frac{1}{5}}}$ (when $x = a$) $= \dfrac{200}{693a}$.

Express in the form of continued fractions:

(40.) i. $\dfrac{67}{68}$; ii. $\dfrac{365}{224}$; iii. $\dfrac{217}{764}$; iv. $\dfrac{84}{227}$; v. $\dfrac{314159}{100000}$.

Find the quotients and convergents for the following fractions

(41.) i. $\dfrac{251}{764}$; ii. $\dfrac{182}{25}$; iii. $\dfrac{1769}{5537}$; iv. $\dfrac{1051}{329}$.

(42.) i. $\sqrt{28}$; ii. $\sqrt{31}$; iii. $\sqrt{45}$; iv. $\sqrt{50}$; v. $\sqrt{17}$.

XXI.

COMPOUND INTEREST AND ANNUITIES.

(1.) Find the compound interest of 1600*l.* for 9 years at 5 per cent. per annum, the interest being payable yearly.

(2.) Find the amount of an annuity of 1712*l.* per annum, payable half-yearly for 9 years, allowing compound interest at 6 per cent. per annum.

(3.) In what time will a given sum treble itself, at 3 per cent. per annum, compound interest, payable half-yearly?

(4.) In how many years will 2653*l.* 7*s.* 6*d.* invested at $3\frac{1}{2}$ per cent. per annum, compound interest, payable quarterly, amount to 8327*l.* 18*s.* 1·104*d.*?

(5.) What is the present value of an annuity of 140*l.* per annum, payable quarterly for five years, allowing compound interest at 5 per cent. per annum?

(6.) What is the discount on 200*l.* due 3 years hence, at $4\frac{1}{2}$ per cent. per annum, compound interest?

(7.) A person puts out 50*l.* at 4 per cent. per annum, compound interest, and adds to his capital at the end of every year a sum equal to the $\frac{1}{8}$ part of the interest for that year; find the amount at the end of 20 years.

(8.) A sum of 7000*l.* is left for three children, A, B, and C, in such a manner that at the end of 7, 9, and 12 years, when they respectively will come of age, they are to receive equal sums; required the present values of each share at 4 per cent. per annum, compound interest.

(9.) A debt of 1000*l.* accumulating at 4 per cent. per annum, compound interest, is discharged in n years by annual payments of 83*l.* 6*s.* 8*d.*; find the value of n.

(10.) A banker borrows money at $3\frac{1}{2}$ per cent. per annum, and pays the interest at the end of the year; he lends it out at the rate of 5 per cent. per annum, but receives the interest quarterly, and by this means gains 400*l.* a year; how much does he borrow?

(11.) Find the present value of a deferred annuity of 500*l.* to commence after the expiration of 5 years, and then to continue for 20 years, allowing compound interest at 5 per cent.

(12.) If a lease for $55\frac{1}{4}$ years cost 100*l.*, what annual rent ought the purchaser to receive, that he may get $5\frac{1}{4}$ per cent. for his money?

(13.) What is the present worth of an annuity of 850*l.* at 5 per cent. per annum, payable quarterly, for 12 years, at compound interest?

(14.) An annuity of 100*l.* for 21 years is sold for 1100*l.*; what was the rate of interest allowed to the purchaser?

(15.) What is the difference between the value of a freehold estate, or perpetual annuity of 200*l.* per annum, and that of a leasehold estate of 200*l.* per annum, to continue 60 years?

(16.) What sum ought to be paid for the reversion of an annuity of 100*l.* for 14 years after the next 7, that the purchaser may make 5 per cent. on his money?

(17.) If a lease of 65½ years be purchased for 500*l.*, what rent ought to be received that the purchaser may make 7 per cent. per annum on his money?

(18.) The sum of 518*l.* 6*s.* being placed out at compound interest for 3 years, amounts to 600*l.*; find the rate of interest.

(19.) Suppose a person to place out annually the sum of 20*l.* for 40 successive years, what would the whole amount to at the end of that time, at 5 per cent. compound interest?

(20.) A person purchases the reversion of an estate after 12 years for 2000*l.*; what rent ought he to receive, that he may realise 6 per cent. per annum on his money?

(21.) Find the present value of an annuity of 100*l.*, to continue 5 years, allowing compound interest at 5 per cent.

XXII.

SCALES OF NOTATION.

(1.) Express 1828 in the septenary scale.

(2.) Transform the number 4321 from the quinary to the septenary scale.

(3.) Transform the number 2304 from the quinary to the undenary scale.

(4.) Transform 1756 and 345 from the common to the duodenary scale, and multiply the results together.

(5.) Divide 29*t*96580 by 2*tt*9 in the duodenary scale.

(6.) Divide 14332216 by 6541 in the septenary scale.

(7.) Express 29 and 49 in the octenary scale; multiply the results together, and express the product in the denary scale.

(8.) Given 468 and 701, numbers in the undenary scale; multiply them together in that scale, and express the product in the denary scale.

(9.) Transform 23784 and 587 from a system whose local value is 9 into one whose local value is 12, and multiply them together in this last system.

(10.) What are (in common notation) the greatest and least numbers which can be expressed by 4 figures when the local value is 6?

(11.) Express 7631 in the scales in which the local values are 2 and 20.

(12.) Extract the square root of 32*e*75721 in the duodenary scale, and express the root in the common system.

(13.) Extract the square root of 25400544 in the senary scale.

(14.) Transform 26.5 into the quaternary scale.

(15.) Find the value of $(1.23)^2$ in the septenary scale, and transform 1.23 into the denary scale.

(16.) Divide 511173.44 by .675 in the octenary scale.

(17.) Any number is divisible by 4, if the last two digits be divisible by 4.

(18.) Any number is divisible by 8, if the last 3 digits be divisible by 8.

(19.) Transform 13.454 from the octenary to the quaternary scale.

(20.) How may the number 2304 in the quinary scale be represented by terms of the series 1, 3, 3^2, 3^3, &c.

(21.) What terms of the series 1 lb., 3 lbs., 3^2 lbs., &c. must be selected to weigh 1319 lbs.?

(22.) How may the series of weights 1, 3, 3^2, &c. be employed to weigh 1000 lbs.?

(23.) How may the series of weights 1, 5, 5^2, &c. be employed to weigh 271 lbs., supposing we have two of each kind?

(24.) If any number a multiple of 11, and a number consisting of the same digits in an inverted order, be each divided by 11, the sum of the digits in the two quotients are equal.

(25.) If the sum of the odd digits in any number be $11m+p$, and of the even $11n+p$, prove that this number, being successively divided by 11 and by 9, leaves the same remainder as $m+n+p$ when divided by 9.

(26.) Any number of 4 places is divisible by 7, if the first and last digits be the same, and the digit in the place of the hundreds be double that in the place of the tens.

(27.) If the number expressed by the last n digits of a number be divisible by 2^n, the number itself is divisible by 2^n.

(28.) The square of any number of digits less than ten, each of which is unity, will, when reckoned from either end, form the same arithmetic series, whose common difference is unity, and greatest term the number of digits in the root.

(29.) In a set of weights of 1, n, n^2, n^3, &c. pounds, how many of each sort will it be necessary to have, in order to weigh any number of pounds not exceeding the sum of the weights, supposing that all the weights used are put in the same scale?

XXIII.

PROPERTIES OF NUMBERS.

(1.) Shew that n^3 divided by 4, if n be one of the natural numbers, cannot have 2 for a remainder.

(2.) If n be a whole number, shew that $n^3 + 5n$ is divisible by 6, and $n(n^2 - 1)(n^2 - 4)$ by 120.

(3.) If n be a whole number, prove that $n.(n-1)(n-2)(n-3)$, is divisible by 24.

(4.) If n be any odd square number greater than 1, prove that $(n+3) \times (n+7)$ is divisible by 32.

(5.) If n be an odd number prime to 5, $n^4 - 1$ is divisible by 80.

(6.) If n be any odd number > than 1, $\dfrac{n(n^4-1)}{48}$ is an integer.

(7.) If n be any even number, $\dfrac{n(n^2 + 20)}{48}$ is an integer.

(8.) If n be any prime number greater than 3, $\dfrac{n^2 - 1}{24}$ is an integer.

(9.) If n be any odd number > than 1, $\dfrac{n^2 - 1}{8}$ is an integer.

(10.) If n be any even number greater than 2, $n(n^2 - 4)$ is divisible by 48.

(11.) The difference of the squares of any two odd numbers is divisible by 8; and the difference of the squares of any two prime numbers of which the less exceeds 5, is divisible by 24.

(12.) If to the square of any number not divisible by 3, the number 2 be added, the result is divisible by 3.

(13.) The square of every number except 3 and its multiples, is of the form $3m+1$.

(14.) If $a^3 - b^3$ be divisible by 3, then $(a \pm k)^3 - (b \pm k)^3$ will be also divisible by 3.

(15.) If an odd and even square number be added together, and the sum be also a square number, the even square is a multiple of 16.

(16.) Every prime number of the form $4m+1$ is the sum of two squares.

(17.) If n be any whole number, one of the three, n^2, n^2+1, $n^2 - 1$, is divisible by 3.

(18.) If n be any prime number greater than 3, $\frac{n^2-1}{24}$ is an integer.

(19.) No number having 9 for its last two digits can be a square.

(20.) If any number which is a perfect square be divided by 3, it can never leave 2 for a remainder.

(21.) If the number 2 be divided into any two parts, the difference of the parts equals the difference of the numbers formed by adding each to the square of the other.

(22. Find the number and sum of the divisors of 2160.

(23) Decompose 831600 into its prime factors; find the number and sum of its divisors; and the least number by which it must be multiplied so as to become a perfect cube.

(24.) What is the least number by which 2205 must be multiplied, so as to become a perfect cube; and what is the sum of its divisors?

(25.) If m be a prime number, and a and b integers not divisible by m, then will $a^{m-1} - b^{m-1}$ be a multiple of m.

26.) The sum of any number of prime numbers in arithmetical progression is a composite number.

XXIV.

EXAMPLES IN THE THEORY OF EQUATIONS.

Form the equations whose roots are :—

(1.) 3 and 4 : 5 and -3, and $\pm\frac{1}{3}$, $\pm\frac{1}{5}$.

(2.) 1, $\dfrac{-1\pm\sqrt{-3}}{2}$, and -1, -3, 3, $\dfrac{1\pm\sqrt{-3}}{2}$.

(3.) 4, -1, $\frac{1}{2}(-3\pm\sqrt{-31})$, and 1, -1, $\pm\frac{1}{2}$, ± 2.

(4.) 2, $-\frac{3}{2}$, $\dfrac{1\pm\sqrt{-43}}{4}$, and 1, $3\pm\sqrt{-\frac{3}{4}}$.

Find the roots of the equation :—

(5.) $x^3-11x^2+37x-35=0$, of which one root is $3+\sqrt{2}$.

(6.) $x^3-10x^2+4x+8=0$, of which one root is $1+\sqrt{3}$.

(7.) $x^4-5x^3-x+5=0$, of which a root is 5.

(8.) $x^3-7x^2+16x-12=0$, of which two roots are equal.

(9.) $x^3-12x^2+43x-42=0$, of which one root is $3+\sqrt{2}$.

(10.) $x^3-12x^2+39x-28=0$, of which the roots form an arithmetic progression.

(11.) $x^4+13x^3+33x^2+31x+10=0$, of which three roots are equal.

(12.) $x^4+x^3-8x^2-16x-8=0$, of which one root is $1-\sqrt{5}$.

(13.) $x^3-9x^2+23x-15=0$, of which the roots are in an arith. prog.

(14.) $x^3-6x^2+11x-6$, of which the roots form an arith. prog.

(15.) $x^4-3x^3-14x^2+48x-32=0$, of which two roots are 1 and 2.

Of which the roots form a geometric progression :—

(16.) $x^3-7x^2+14x-8=0$.

(17.) $x^3-13x^2+39x-27=0$.

(18.) $x^3-26x^2+156x-216$.

(19.) $x^4-15x^3+70x^2-120x+64=0$.

Of which two roots are equal :—

(20.) $x^3-8x^2+27x-18=0$.

(21.) $x^3+8x^2+20x+16=0$.

Of which the roots are in a harmonic progression :—

(22.) $x^4 - 11x^2 + 36x - 36 = 0.$

(23.) $8x^3 - 6x^2 - 3x + 1 = 0.$

(24.) $48x^3 - 44x^2 + 12x - 1 = 0.$

(25.) $x^3 - 13x^2 + 54x - 72 = 0.$

Form into equations having integral coefficients—

(26.) $x^4 - \frac{5x^3}{6} + \frac{5x^2}{12} - \frac{7x}{150} - \frac{13}{900} = 0;$

(27.) $x^2 + \frac{2x}{3} - \frac{56}{3} = 0;$

(28.) $x^3 - \frac{15x^2}{4} - \frac{3x}{2} - \frac{15}{4} = 0$

whose roots shall be double the roots of—

(29.) $3x^3 - 5x^2 + 2x - 7 = 0.$

whose roots shall be less by 5 than those of—

(30.) $x^4 - 19x^3 + 117x + 1145 = 0.$

whose roots shall be greater by unity than those of—

(31.) $x^5 + 5x^4 + x^3 - 16x^2 - 20x - 16 = 0.$

having integral coefficients :—

(32.) $x^3 - \frac{4x^2}{3} - \frac{3x}{8} + \frac{5}{72} = 0;$

whose roots shall be 5 times those of—

(33.) $x^4 + 2x^3 - 7x - 1 = 0.$

whose roots shall be 3 times those of—

(34.) $x^4 + 7x^2 - 4x + 3 = 0.$

whose roots shall be each less by 3 than those of—

(35.) $x^3 - 27x - 36 = 0.$

whose roots shall be less by 3 than those of—

(36.) $x^4 - 27x^2 - 14x + 120 = 0.$

whose roots shall be 3 greater and 3 less than those of—

(37.) $x^3 - 2x^2 - x + 2 = 0.$

Take away the second term from the equations—

(38.) $x^3 - 6x^2 + 12x - 19 = 0.$

(39.) $x^3 - 6x^2 + 5 = 0.$

Tranform into an equation whose roots shall be less by unity—

(40.) $2x^4 - 13x^2 + 10x - 19 = 0.$

Transform—

(41.) $19x^4 - 22x^3 - 35x^2 - 16x - 2 = 0$ into an equation whose roots shall be less by 3.

(42.) $3x^4 - 13x^3 + 7x^2 - 8x - 9$ into an equation whose roots shall be less by $\frac{1}{3}$.

(43.) $2x^3 - 5x^2 + 7x - 12 = 0$ into an equation whose roots shall be 3 times as great.

Transform into equations wanting the second term—

(44.) $x^3 - 6x^2 + 4x - 7 = 0.$

(45.) $x^3 - 6x^2 - 10 = 0.$

(46.) $x^3 - 6x^2 + 11x - 6 = 0.$

(47.) $x^4 - 4x^3 - 8x^2 + 32 = 0.$

(48.) $x^3 + 3x^2 + 9x - 13 = 0.$

Solve the following reciprocal or recurring equations—

(49.) $x^4 + 5x^3 + 2x^2 + 5x + 1 = 0.$

(50.) $x^9-7x^8+15x^7-13x^6+8x^5+8x^4-13x^3+15x^2-7x+1 = 0$, of which -1 is a root.

(51.) $x^4 - \frac{5x^3}{2} + 2x^2 - \frac{5x}{2} + 1 = 0.$

(52.) $2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0.$

Find the superior limit of the positive roots of the equations—

(53.) $x^3 + 5x^2 - 7x + 1 = 0.$

(54.) $x^5 + x^4 + x^2 - 25x - 36 = 0.$

(55.) $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0.$

(56.) $6x^3 - 43x^2 + 79x - 12 = 0.$

(57.) $x^3 - 112x + 448x = 0.$

Approximate to one of the roots of the following equations:—

(58.) $2x^3 - 3x - 6 = 0.$ (59.) $x^3 - 7x - 7 = 0.$

(60.) $5x^6 - 13x^3 - 1100 = 0.$ (61.) $x^3 - 5x - 3 = 0.$

Find the roots of the following equations—

(62.) $x^3 - 9x + 28 = 0.$ (63.) $x^3 - 6x^2 + 11x - 6 = 0.$

(64.) $x^3 + x^2 + x = 100.$ (65.) $x^4-12x^2+12x-3 = 0.$

(66.) $x^3 - 15x^2 + 63x = 50.$ (67.) $64x^3-48x-9 = .$

XXV.

(1.) Multiply together $x+2\sqrt{xy}+3y$, and $x-2\sqrt{xy}+y$; also $a+b+c$, $-a+b+c$, $a-b+c$, and $a+b-c$.

(2.) Simplify the expression—

$$\frac{x^2-y^2+2x^{\frac{5}{3}}y^{\frac{1}{3}}+2x^{\frac{1}{3}}y^{\frac{5}{3}}}{x^3+y^3}.$$

(3.) Extract the square root of—

$$a^2x^2+4abx+6ac+4b^2+\frac{12bc}{x}+\frac{9c^2}{x^2}.$$

(4.) Solve the equations—

(i.) $\frac{x}{8}-\frac{2(x-1)}{5}=\frac{3x-4}{15}+\frac{x}{12}$.

(ii.) $7x^2-11x=6$.

(iii.) $x^4+1=x(x^2-x\sqrt{2}+1)$.

(iv.) $\frac{x+y-\sqrt{x^2+y^2}}{x+y+\sqrt{x^2+y^2}}=\frac{2x}{a}$, and $\frac{x}{y}=\sqrt{\frac{a+x}{a-y}}$.

(5.) Find three numbers such that the differences between each and the sum of the other two, and their product, are in the ratios 1 : 3 : 5 :: 192.

(6.) Prove that—

(i.) The difference of the squares of any two odd numbers is divisible by 8;

(ii.) Every square is the sum of as many consecutive odd numbers as there are units in the root.

(7.) How many terms are there in the expansion of $(3x-8y)^6$? Write down the third term.

(8.) Prove the formulæ for the summation of arithmetical and geometric series; and sum the following:—

$7+3-1-..$ to n terms;

$1-\frac{1}{3}+\frac{1}{9}-..$ to infinity.

(9.) Find the least positive integer solution of $13x-9y=3$.

(10.) Determine the number of permutations of n letters taken r together, noticing the case in which $n=r$ and certain letters recur. Ex.: Sebastopol.

(11.) Reduce to their simplest form the expressions—

(*a.*) $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y}$, and $\dfrac{x+y}{x \mp y} - \dfrac{x-y}{x+y}$.

(*b.*) $(x-3y)\cdot(x+y) - (x-2y)(x+3y) + (x-y)(x+5y) - x^2 + 2y^2$.

(12.) Find the greatest common measure of $x^4+x^3y-2x^2y^2+xy^3-y^4$, and $x^3-6x^2y+11xy^2-6y^3$.

(13.) Write down the expansion of $(x+y)^6$. How many terms are there in the expansion of $(x+y)^{100}$?

(14.) Extract the square roots of $14+6\sqrt{5}$, and $1+4\sqrt{-3}$.

(15.) Solve the equations—

(*a*) $\dfrac{x-2}{11} + \dfrac{3x-22}{17} + \dfrac{x-3}{5} = 4$.

(*b*) $x+13y-\dfrac{8x-12y}{5} = 29$, and $7x-2y-\dfrac{2x-y}{4} = 16$.

(*c*) $3x^2-\dfrac{x}{7} = 146$.

(*d*) $2x^2+3x+\sqrt{2x^2+3x}-16 = 74$.

(16.) A traveller sets out to walk from A to B at 4 miles an hour; one hour later, another traveller sets out from B towards A at $3\frac{1}{2}$ miles an hour: when they meet, the first has walked 6 miles more than the second. Find the distance from A to B.

(17.) Sum the arithmetical progression $1+5+9+\ldots.$ to 15 terms, and the geometric progression $1+\frac{1}{5}+\frac{1}{25}+\ldots.$ to n terms and *ad infinitum*.

XXVI.

Solve the following equations:—

(1.) $\dfrac{x-3}{x+2} = \dfrac{x-3}{2x-1} + \dfrac{1}{2}$.

(2.) $x + \dfrac{1}{2}y + \dfrac{1}{3}z = 32$, $\dfrac{x}{3} + \dfrac{y}{4} + \dfrac{z}{5} = 15$, $\dfrac{x}{4} + \dfrac{y}{5} + \dfrac{z}{6} = 12$.

(3.) $x^4 + y^2 = 49 + 2x^2y$, $x^4 + y^4 - x^2 = 20 + (2x^2 - 1)y^2$.

(4.) Reduce to its simplest form—

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}.$$

(5.) If $\frac{a}{b}$ and $\frac{a'}{b'}$ be two unequal fractions, prove that $\frac{a+a'}{b+b'}$ is intermediate in value between them.

(6.) If the same transformation banishes both the second and third terms from the cubic equations $x^3-px^2+qx-r=0$, what relations must exist between the coefficients?

(7.) Solve the equations $3x+2y+z=23$, $5x+2y+4z=46$, $10x+5y+4z=75$; and $2x-y=2$, $8x^3-y^3=98$.

(8.) A railway train running from London to X, encounters an accident on the way, in consequence of which its rate is reduced to two-thirds of what it was at first, and it is 2 hrs. 5 m. late; if the accident had happened 18 miles nearer X, it would have been only 1 hr. 50 m. late. Find the distance between the two towns.

(9.) Extract, by Horner's method, the cube root of 20, and find the root of the equation $x^3+2x-30=3$ (the integer part of which is 2).

XXVII.

(1.) Perform the operation indicated in the expression $(a-b)(c-d)$, and deduce the "Rule of Signs."

Multiply $a^{\frac{3}{4}}x^{\frac{2}{3}}-b^{\frac{2}{5}}y^{\frac{1}{6}}$ by $a^{\frac{1}{4}}x^{\frac{1}{3}}-b^{\frac{3}{5}}y^{\frac{5}{6}}$.

Simplify $\frac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1}$.

(2.) Solve the following equations:—

(i.) $\frac{x}{x+1}+\frac{x+1}{x+2}=\frac{x-2}{x-1}+\frac{x-1}{x}$.

(ii.) $2x^2+6x=5-\sqrt{x^2+3x-1}$.

(iii.) $x^2+y^2=20$, $x+y=\sqrt{2xy}+2$.

G

(3.) A railroad runs from A to C. A goods train starts from A at 12 o'clock, and a passenger train at 1 o'clock. After going two-thirds of the distance the goods train breaks down, and can only travel at three-fourths of its former speed. At 40 minutes past 2 o'clock a collision occurs 10 miles from C. The rate of the passenger train being double of the diminished speed of the goods train; find the distance from A to C, and the rates of the trains.

(4.) Prove the Binomial Theorem for a positive integral index.

(5.) Find the first four terms of $\left(1 - \frac{x}{8}\right)^{-8}$ and write down the factors of the 15th term.

(6.) If $1, a, \beta, \gamma \ldots.$ be the coefficients in order in the expansion of $(1 + x)^n$, show that if n be a positive integer greater than unity, $1 - 2a + 3\beta - 4\gamma + \ldots. = 0$.

(7.) If one solution to an indeterminate equation $ax + by = c$ be known, give formulæ for finding out the rest.

In how many different ways is it possible to pay 2*l.* 3*s.* 6*d* in half-crowns and shillings?

(8.) From the expansion of $(x + a)^2$, deduce the ordinary rule for "completing the square" in the solution of a quadratic equation. Solve the equations—

(i.) $2x^2 - 1 = 5x + 2$. (ii.) $2x^3 - 3x^2 + 5 = 0$.

(iii.) $2\sqrt{x^2 + x} + 2x = 1 - \sqrt{x} - \sqrt{1 + x}$.

(iv.) The difference of two numbers is 3, and the difference of their cubes is 279; find the numbers.

(9.) Explain how to find the sum of a series of n terms in arithmetical progression, whose first term is a, and last term l.

Sum the series, $6, -2, \frac{2}{3}, -\frac{2}{9}$, &c. to infinity.

(10.) State the algebraical definition of proportion.

If $a : b :: c : d$, prove that $a : a-b :: c : c-d$,

and $\frac{1}{a} - \frac{1}{2b} - \frac{1}{3c} + \frac{1}{4d} = \frac{1}{ad}\left(\frac{a}{4} - \frac{b}{3} - \frac{c}{2} + d\right)$.

(11.) If A $\propto$ B when C is constant; and A $\propto$ C when B is constant; then A $\propto$ BC when B and C both vary.

Show the application of this in the following example:—A beam 56 feet long, $2\frac{3}{4}$ broad, $1\frac{1}{3}$ thick, costs 11*l.*; what will be the cost of a beam 154 feet long, with a uniform section of $1\frac{1}{8}$ square feet?

XXVIII.

(1.) What signs are used in Algebra to express Addition, Subtraction, Multiplication, and Division?

How does it appear $ab = ba$, and $a \div b = \frac{a}{b}$.

(2.) (i.) Subtract $a - \frac{b}{2} + c$ from the sum of $a + \frac{b}{2}$ and $\frac{a}{3} + c$.

(ii.) Reduce $\frac{a}{x} + \frac{2a^2 - x^2}{a^2 - x^2} - \frac{a}{a + x}$.

(iii.) Multiply $a^2 - ab - ac - bc + b^2 + c^2$ by $a + b + c$.

(iv.) Divide $42xy^3$ by $-7x^{\frac{1}{2}}y^{\frac{3}{2}}$.

(3.) (i.) If $x = \frac{a - b}{a + b}$ and $y = \frac{a + b}{a - b}$, prove that

$$\frac{x + y}{x - y} = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right).$$

(ii.) Reduce to its lowest terms $\frac{3x^2 - x - 4}{6x^2 + 7x - 20}$.

(4.) What are the rules for transposing an algebraical quantity from one side of an equation to another?

Solve the equations:—

(i.) $\frac{2(x - 4)}{3} + 5x = x - \frac{1}{3}$.

(ii.) $\frac{2x - 72}{2} + \frac{x}{15} - \frac{x - 25}{5} = x - 44$.

(iii.) $\frac{3x - 1}{4} + \frac{7y - 2}{3} = 22$, and $x = 3y$.

(5.) There are two sorts of wine; one cost 5*s.* a quart, and the other 3*s.* a quart. How much of each must be taken, to mix a quart which shall be worth 3*s.* 6*d.*?

(6.) Solve the equations—

(i.) $\frac{xy}{x + y} = 1$, $\frac{xz}{x + z} = 2$. $\frac{yz}{y + z} = 3$.

(ii.) $x^4 + y^4 = 337$, and $x + y = 7$.

(7.) A garrison besieged loses 5 men on the first day, 10 on the second, 15 on the third, and so on, and at this rate the garrison would be all destroyed in 50 days. How many men must be introduced at the end of the fifteenth day of the siege, to bring the garrison up to a strength of 8000 men?

(8.) (i.) Prove that $\log 3^5 \times 4^6 = 5 \log 3 + 6 \log 4$.

(ii.) Find by the tables $(\cdot 0874)^{\frac{1}{3}}$.

(iii.) A fourth proportional to 3^5, 2^7, 15^8.

(iv.) The sum of 7 terms of the series $7 + 21 + 63 +$ &c.

XXIX.

(1.) What are *like quantities* in algebra? Point out which of the following are *like*, or may be made so, and which *unlike*:—

$$7a^2b, \quad 3ba^2, \quad 4a^2b^2, \quad 4a^3b, \quad \frac{8a^3b^2}{2ab}.$$

(2. Perform the operations indicated in the following examples:—

(i.) $4a^2 + 5b^2 - [3a^2 - (2a^2 + b^2)]$.

(ii.) $\dfrac{4}{(x+1)^2} - \dfrac{5}{(x^2-1)} + \dfrac{x}{x-1}$.

(iii.) $\left(x^2 - \dfrac{xy}{2} + y^2\right) \times \left(x^2 + \dfrac{xy}{2} + y^2\right)$

(iv.) $(x^4 + x^2y^2 + y^4) \div (x^2 - xy + y^2)$.

(v.) $\sqrt{a^4 - 4a^2b^2 + 2a^2c^2 - 4b^2c^2 + 4b^4 + c^4}$.

(3.) Arrange the product $(x-a) \times (x-b) \times (x-c)$ according to powers of x, and then find its value, if $a = b = c$.

Find the greatest common measure of $4x^2 - 9y^2$ and $6x^2 - 5xy - 6y^2$.

(4.) Explain the principle upon which equations may be cleared of fractions.

Solve the following equations:—

(i.) $\dfrac{x+1}{2} - \dfrac{5-x}{4} = 14 - \dfrac{x+2}{3}$.

(ii.) $\dfrac{3y-x}{3} - \dfrac{3x-y}{5} = 10, \quad \dfrac{y+x}{y-x} = 7$.

(5.) Two regiments are in strength as the numbers 5 : 4, but if 200 men be transferred from the smaller regiment to the greater, their respective strengths will be as 2 : 1. Find the number in each regiment.

(6.) Solve the following equations:—

(i.) $16(1+x^3) = 7(1+x)^3$.

(ii.) $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}} = 133$, and $x^{\frac{1}{3}}-x^{\frac{1}{6}}y^{\frac{1}{6}}+y^{\frac{1}{3}} = 7$.

If $\dfrac{c-a}{x-z} = \dfrac{c-b}{y-z} = \dfrac{a+b+c}{2(x+y+z)}$, prove that

$$\frac{c}{x+y} = \frac{a}{y+z} = \frac{b}{x+z}.$$

(7.) Prove that the sum of the squares of the first (n) natural numbers $(1^2+2^2+3^2+\ldots\ldots+n^2)$

$$= \frac{n.(n+1).(2n+1)}{6}.$$

Hence find the number of balls in an incomplete square pile, a side of the base having 30 balls, and a side of the top 11. Show why $\dfrac{n.(n+1).(2n+1)}{6}$ must always be a whole number, when (n) is a whole number.

XXX.

(1.) Find the value of $\sqrt{x^2-2xy+y^2} \times \sqrt{x^2+2xy+y^2}$, when $x = \frac{1}{4}$, and $y = \frac{1}{8}$.

(2.) Multiply $x^2 - \dfrac{2x}{3} + \dfrac{3}{4}$ by $x^2 + \dfrac{2x}{3} + \dfrac{3}{4}$, and divide $a^4 - 81$ by $a-3$.

(3.) Explain the laws for finding the Greatest Common Measure and Least Common Multiple of two or more quantities; and find the G. C. M. of x^2-2x-3, $x^2-7x+12$, and x^2-x-6, and the L. C. M. of $2x-1$, $4x^2-1$, and $4x^2+1$.

(4.) Define a fraction; and simplify the following quantities:—

(i.) $\dfrac{a}{5x}+\dfrac{3a}{4x}-\dfrac{7a}{10x}$.

(ii.) $\dfrac{4m-3n}{3(1-n)}-\dfrac{m+3n}{3(1-n)}+\dfrac{2n}{1-n}$.

(iii.) $\dfrac{a-b}{ab}-\dfrac{a-c}{ac}+\dfrac{b-c}{bc}$.

(iv.) $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^3-x^2y}{y^3-x^2y}$.

(v.) $\dfrac{a+b}{ab}(a^2+b^2-c^2)+\dfrac{b+c}{bc}(b^2+c^2-a^2)+\dfrac{a+c}{ac}(a^2+c^2-b^2)$.

(5.) Investigate the laws for finding the Square and Cube Root of any quantity; and find the Square Root of

$$x^4 - 2x^3 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16},$$

and the Cube Root of $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$.

(6.) Define a surd; and simplify the following quantities:—

(i.) $2\sqrt{8} - 7\sqrt{18} + 5\sqrt{72} - \sqrt{50}$.

(ii.) $5\sqrt{3} \times 7\sqrt{\frac{8}{147}} \times \sqrt{\frac{2}{3}}$.

(iii.) $\sqrt[4]{29 + 4\sqrt{30}} \times \sqrt[4]{29 - 4\sqrt{30}}$. (iv.) $\sqrt{13 + 2\sqrt{42}}$.

(7.) Solve the following equations:—

(i.) $\frac{x+6}{4} - \frac{16-3x}{12} = \frac{25}{6}$. (ii.) $\frac{6x+8}{2x+1} - \frac{2x+38}{x+12} = 1$.

(iii.) $3x + 5y = 8$ and $4x + 3y = 7$. (iv.) $(x+2)^2 - (x+2) = 6$.

(v.) $9x + 4 + 2x\sqrt{9x+4} = 15x^2$. (vi.) $a + x + \sqrt{2ax + x^2} = b$.

(vii.) $mqx^2 - mnx + pqx = np$. (viii.) $x^4 + x^3 + x + 1 = 4x^2$.

(ix.) $\frac{x - \sqrt{x^2 - y^2}}{x + \sqrt{x^2 - y^2}} - x = 0$, and $\frac{x}{y} - \sqrt{\frac{1+x}{1-y}} = 0$.

(8.) Prove the laws for the summation of an Arithmetic Series; and find the sum of $1 + 3 + 5 +$ &c. to 10 and n terms, and $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} +$ &c. to 20 terms.

(9.) Find the nth term and the sum of n terms of the Series, $1 + 2 + 4 + 8 +$ &c., and the sixth term and the sum of 6 terms of the Series, $1 - 3 + 3^2 -$ &c.

(10.) If the number of permutations of $2n+1$ things taken $n-1$ together, be to the number of $2n-1$ things taken n together, as 13 : 4, find n.

(11.) If a person divide £5. among 36 persons, male and female, giving to each male 3*s*., and to each female 2*s*. 6*d*.; how many were there of each?

(12.) Prove the Binomial Theorem; and expand $(a + x)^6$ and $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^8$.

(13.) Resolve $\frac{3x-1}{x^2(x+1)^2}$ into its partial fractions.

(14.) Find the value of

$$\sqrt{\frac{1+x}{1-y}}+\sqrt{\frac{2y\,(1-x^2)}{1-y^2}}+\sqrt{x^2-4xy+4y^2},$$

when $x=\frac{1}{4}$, and $y=\frac{1}{8}$.

(15.) Multiply $a^2+b^2+c^2+d^2$ by $a^2+b^2-c^2-d^2$. Divide $x^4+a^2x^2+a^4$ by x^2-ax+a^2, and $mpx^3+(mq-nq)\,x^2-(mr+nq)\,x-nr$ by $mx-n$.

(16.) Prove the laws for finding the *greatest common measure* and *least common multiple* of two or more quantities, and apply it to find the G. C. M. of $3a^4-a^2b^2-2b^4$ and $10a^4+15a^3b-10a^2b^2-15ab^3$, and the L. C. M. of x^2-2x+1, x^3-3x^2+3x-1, and x^3-x^2-x+1.

(17.) Simplify the following fractions:—

(i.) $\dfrac{1}{2\,(x+1)}+\dfrac{9}{2\,(x+3)}-\dfrac{x^2}{(x+1)\,(x+2)\,(x+3)}$.

(ii.) $\dfrac{2}{(x+1)^2}+\dfrac{8}{x+1}-\dfrac{8x+1}{x^2+x+1}-\dfrac{x^2}{(x^2+x+1)\,(x+1)^2}$.

(iii.) $\dfrac{1}{(a-b)\ (a-c)\ (x+a)}-\dfrac{1}{(b-a)\ (c-b)\ (x+b)}+\dfrac{1}{(c-a)\ (c-b)\ (x+c)}$.

(18.) Define a Surd, and simplify the following expressions :—

(i.) $\frac{1}{2}\sqrt{32}-\frac{7}{3}\sqrt{162}+\frac{5}{2}\sqrt{288}-\frac{1}{5}\sqrt{200}$.

(ii.) $\dfrac{xy}{x-y}+\sqrt{\dfrac{x^2y^3}{(x-y)^2}+\dfrac{x^2y}{x-y}}-\dfrac{x\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

(19.) Prove the method of finding the square root of a quadratic surd, and find the square root of $28-10\sqrt{3}$ and $-18\sqrt{-1}$.

(20.) Prove the law for finding the sum of an arithmetic series, and find the sum of $\frac{2}{3}+\frac{7}{15}+$ &c. to 6 terms; and the number of terms when the first term is 3, the difference 5, the sum 1010.

(21.) Prove the law for the expansion of a binomial, and expand $(a+2b)^6$ and $(a^2-x^2)^{-\frac{1}{2}}$

(22.) Solve the following equations:—

(i.) $\dfrac{16x+11}{28} + \dfrac{11x+21}{6x+14} = \dfrac{8x+30}{14}.$

(ii.) $x^2-2x+6\sqrt{x^2-2x+5} = 11.$

(iii.) $\dfrac{m}{x} + \dfrac{n}{y} = a$, and $\dfrac{n}{x} + \dfrac{m}{y} = b.$

(iv.) $(x^2+y^2)^2+4xy(x+y)^2 = 1396$, and $x-y=4.$

(v.) $x+y+z=14$, $x^2+y^2+z^2=84$, and $xz=y^2.$

(23.) A, B, C, reaped a field in a certain time; A alone could do it in 12 hours more, B in twice the time that A could, and C in three times as long as B. What time did it take all of them?

(24.) Show that $x^4+px^3+qx^2+rx-s^2$ can be resolved into rational quadratic factors, if $s^2 = \dfrac{r^2}{p^2-4q}$; and thence solve the equation $x^4-6x^3+5x^2+8x-4=0.$

(25.) Resolve $\dfrac{x^2+1}{x^4+x^2+1}$ into partial fractions.

(26.) Find the fourth term of $(a^2+b^2\sqrt{-1})^{-\frac{2}{3}}$; and the coefficient of x^4 in $(2-5x+7x^2)^5$.

(27.) If S_1, S_2, S_3, &c., denote the sums of an infinite number of infinite decreasing geometrical progressions, whose first terms are a, a^2, a^3, &c., and common ratio r, $2r$, $3r$, &c., prove that

$$S_1+S_2+S_3+\ldots\ldots S_n = \frac{n(n+1)}{2}\left(\frac{r^n-1}{r-1}\right)a.$$

XXXI.

(1.) A railway train, after travelling for one hour, meets with an accident which delays it one hour, after which it proceeds at three-fifths of its former rate, and arrives at the Terminus three hours behind time; had the accident occurred 50 miles further on, the train would have arrived $1\frac{1}{3}$ hour sooner. Required the length of the line.

(2.) Multiply together $a+x$, $b+x$, and $c+x$, and resolve x^6-a^6 into four factors. Explain the meaning of x^n, when n is (1) fractional and (2) negative.

(3.) Define the base of a system of logarithms. What is meant by the characteristic and the mantissa? Show that in the common system, the characteristic of the logarithm of any number can be determined by inspection.

(4.) Given $\log 2 = \cdot 301030$, $\log 3 = \cdot 477121$. Find $\log 24$, $\log 5\cdot 4$, and $\log \cdot 006$.

(5.) Solve the equations—

(α) $\frac{x+6}{2} - \frac{x-7}{3} = 2x - 13.$

(β) $7x + 11y = 57$, and $13x - 21y = 23$.

(γ) $x^2 + xy + y^2 = 7$, and $x - y = 1$.

What condition must be fulfilled in order that the two roots of the quadratic equation $ax^2 + bx + c = 0$, may be equal? How does it appear that a quadratic equation cannot have more than two roots?

(6.) If a napoleon be worth 15*s.* $10\frac{1}{2}$*d.*, find the least exact number of napoleons that must be given for an exact number of English sovereigns. What is the least number of napoleons that may be exchanged for an exact number of sovereigns and shillings, so that the number of sovereigns and shillings together shall be the least. Specify the numbers in each case.

(7.) State and examine the rule for "double position." Solve by means of this rule the following problem. A workman was hired for 60 days with the condition that he was to receive 3*s.* per day for every day he worked, and to forfeit one shilling a day for every day he was idle, and at the end of the time he received 6*l.* 12*s.* How many days did he work?

(8.) In any equation $x + \sqrt{y} = a + \sqrt{b}$, which involves rational quantities and quadratic surds, the rational parts on each side are equal, and also the irrational. Show how to find the square root of a binomial, one of whose terms is rational, and the other a quadratic surd. Examine when the square root of an expression of the form $a + \sqrt{b} + \sqrt{c} + \sqrt{d}$ may be found by an analogous method. Extract the square roots of—

(α) $\sqrt{18} + \sqrt{16}$. (β) $6 + \sqrt{8} - \sqrt{12} - \sqrt{24}$.

(9.) If $\frac{a}{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \&c.$, show that

$$\frac{(a^n + a_1{}^n + a_2{}^n + \&c.)^{\frac{1}{n}}}{(b^n + b_1{}^n + b_2{}^n + \&c.)^{\frac{1}{n}}} = \frac{a}{b} = \frac{na + n_1a_1 + n_2a_2 + \&c.}{nb + n_1b_1 + n_2b_2 + \&c.}.$$

If a, b, c, be positive and unequal quantities, prove that $(a + b + c)(ab + ac + bc)$ is greater than $9abc$.

(10.) In the expansion of $(1+x)^n$, show that the coefficient of x is always n, whether n be positive, negative, whole, or fractional. Show that the $(r+1)$th term of the series for $(1+x)^{\frac{1}{2}}$ may be written

$$\frac{1.3.5.7....(2r-3)}{1.2.3....r}(-1)^{r-1}\frac{x^r}{2^r}.$$

Assuming the form of the series for e^x, prove that $\left(1+\frac{x}{n}\right)^n$ has for its limit e^x when n is increased without limit.

(11.) Show how to reduce a whole number or fraction, expressed in the decimal scale of notation, to an equivalent number in any scale having a different radix; and show that in any scale, the radix of which is r, any whole number when divided by $r+1$ will leave the same remainder as the difference between the sum of the digits in the odd and even places leaves when divided by $r+1$. Express 51782·125 in the scale whose radix is 8. Extract the square root of 3445·44 in the scale whose radix is 6.

XXXII.

(1.) State the rule for Multiplication in Algebra (1) of monomials, and (2) of multinomials; and multiply

$$a^3-2a^2x+ax^2-\tfrac{1}{3}x^3 \text{ by } 3a^2-ax+3x^2.$$

(2.) Divide $a^2+(2ac-b^2)x^2+c^2x^4$ by $a-bx+cx^2$.

(3.) Reduce to their simplest forms—(i.) $\dfrac{x^2+3x-4}{x^3+x^2-4x+2}$.

(ii.) $\dfrac{a}{x(a-x)}-\dfrac{x}{a(a-x)}$. (iii.) $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}$.

(4.) Solve the following equations:—

(i.) $\dfrac{2x-5}{6}+\dfrac{6x+3}{4}=5x-17\tfrac{1}{2}$.

(ii.) $\dfrac{x-a}{a-b}-\dfrac{x+a}{a+b}=\dfrac{2ax}{a^2-b^2}$.

(iii.) $\dfrac{x-1}{4}-\dfrac{y-2}{5}=1$, and $x-\dfrac{2y-5}{3}=y-1$.

(5.) A. and B. ran a race, which lasted five minutes: B. had a start of 20 yards; but A. ran 3 yards whilst B. was running 2, and won by 30 yards. What was the length of the course, and the speed of each?

(6.) Find $\sqrt{x^4 - 2ax^3 + 5a^2x^2 - 4a^3x + 4a^4}$, and $\sqrt[3]{46656}$.

(7.) Simplify (i.) $\left[\frac{a^{-2}}{b^{-2}}\left(\frac{a}{b}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}$.

(ii.) $(\sqrt{5}-\sqrt{2})(\sqrt{2}+\sqrt{1})(\sqrt{5}+\sqrt{2})(\sqrt{2}-\sqrt{1}.)$

(8.) Solve the following equations:—

(i.) $3x^2 - 4x = 32$. (ii.) $\frac{x-1}{x+3} + \frac{x-3}{x+1} = 1$.

(iii.) $x^2 + y^2 = \frac{5}{9}$, and $x^2 - xy + y^2 = \frac{1}{4}$.

(9.) Find a number the square of which shall exceed the square of a given number by three-halves of the product of the numbers themselves.

(10.) Sum the following series:—

(i.) $1 + 2\frac{1}{2} + 4 + \ldots$ to 10 terms.

(ii.) $\frac{3}{8} + \frac{1}{4} + \frac{1}{6} + \ldots$ to 6 terms.

(iii.) $\cdot23 + \cdot0023 + \cdot000023 + \ldots$ to infinity.

(11.) Define arithmetical, geometrical, and harmonical series; find the harmonical mean between 2 and 7; and shew that the sum of the first and last terms of an arithmetical series is equal to the sum of any two terms equally distant from the first and last terms respectively.

(12.) If $a : b :: c : d :: e : f$, shew that

(i.) $\frac{a+b}{a-b} = \frac{c+d}{c-d} = \frac{e+f}{e-f}$. (ii.) $\frac{a+c+e}{b+d+f} = \frac{a}{b}$.

(13.) A nnmber of two digits : the number formed by inverting the digits :: 7 : 4. Shew that the first digit : the second digit :: 2 : 1.

(14.) Find the sum of the series—

(i.) $2\frac{1}{2} + 3\frac{3}{4} + 5 + \ldots$ to 12 terms.

(ii.) $15 - 3 + \frac{3}{5} - \ldots$ to infinity.

(iii.) $\frac{a+b}{2} + a + \frac{3a-b}{2} \ldots$ to n terms.

(15.) Shew that $\dfrac{a+b+c+d}{p+q+r+s}$ is greater than the least and less than the greatest of the fractions $\dfrac{a}{p}$, $\dfrac{b}{q}$, $\dfrac{c}{r}$, $\dfrac{d}{s}$, each letter representing a positive quantity.

(16.) Two casks A. and B. were filled with two kinds of sherry, mixed in the cask A. in the proportion of 2 : 7, and in B. in the proportion of 1 : 5. What quantity must be taken from each to form a mixture which shall consist of 2 gallons of the one kind and 9 of the other?

(17.) Three quantities a, b, c are in arithmetic, geometric, or harmonic progression, according as

$$\frac{a-b}{b-c} = \frac{a}{a}, \text{ or } = \frac{a}{b}, \text{ or } = \frac{a}{c}.$$

(18.) The number of combinations of n things taken r together is the same as the number of combinations of n things taken $n-r$ together. How many words can be made of the letters of the word *rotation*?

(19.) In what scale will the common number 5261 be expressed by 40205? What are the greatest and least numbers which can be expressed with five digits in that scale?

(20.) Express $\sqrt{50}$ in the form of a continued fraction.

(21.) A person makes 20 lbs. of tea at 4*s*. 9*d*. by mixing three kinds at 3*s*. 6*d*., 4*s*. 6*d*., and 5*s*. How can this be done, using no fractions of a pound?

(22.) A., B., and C. engage to do a piece of work which would take A., the best workman of the three, nine days to do single-handed. They work altogether for three days, when A. is disabled. In how many days will B. and C. finish the work, B. being only half as good, and C. three-fourths as good a workman as A.?

(23.) Revert the series—

$$y = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \ldots.$$

(24.) Find the number of balls in a triangular pile, each side of the base containing 25 balls.

(25.) Obtain the rationalizing factor of $3^{\frac{1}{2}} - 5^{\frac{2}{3}}$.

(26.) Expand by the binomial theorem—

$$(a-x)^6, \text{ and } (1-2x)^{\frac{5}{2}} \text{ to 6 terms.}$$

ANSWERS.

I.

No. 2. [*Ans.* $\frac{1330}{1261}$.] 3. [$\frac{144}{481}$.] 4. [$\frac{1}{400}$.] 5. [508.]

6. [$\frac{1}{9} + \frac{\sqrt{41}}{3}$.] 7. [$\frac{2426}{15625}$.] 8. [8.] 9. [$-20$]

10. [$4\sqrt{-3\sqrt{3}} - 7 + 4\sqrt{3} + 2\sqrt{-\sqrt{3}}$.] 11. [0.] 12. [81.]

13. [$\frac{11}{10} + \frac{5\sqrt{7}}{8}$.]

III.

No. 1. [*Ans.* $2a + 3x$.] 2. [$x^2 + ax + a^2$.] 3. [$3x^2 + a^2$.]
4. [$x - 7$.] 5. [$x - 3$.] 6. [$x^2 + 7$.] 7. [$5x^2 - 1$.] 8. [$a^2 - x^2$.]
9. [$x^2 - y^2$.] 10. [$a(2ax - 3y^2)$.] 11. [$x - 1$.] 12. [$x + 3$.]
13. [$x^3 + x^2y + xy^2 + y^3$.] 14. [$(x - a)^3$.] 15. [$(a^3 + x^3)(a - x)$.]
16. [$16x^4 - 1$.] 17. [$x^4 - 5x^2 + 4$.] 18. [$8(1 - x^4)(1 - x)$.]
19. [$a^6 - x^6$.] 20. [$x^4 - a^4$] 21. [$x^6 - y^6$] 22. [$(a^4 - b^4)(a^6 - b^6)$.]
23. [$(x - 1)^3 (x + 1)(x^2 + 1)$.]

IV.

No. 1. [*Ans.* $4a^2 + 12ab + 9b^2$, $\frac{a^2}{4} \pm \frac{ab}{3} + \frac{b^2}{9}$, $a^2 - 6ab + 9b^2$.]

2. [$1 + 2x + 3x^2 + 2x^3 + x^4$, $1 + x + \frac{3x^2}{4} + \frac{x^3}{4} + \frac{x^4}{16}$, $\frac{x^4}{16} + \frac{x^3y}{6} + \frac{17x^2y^2}{72} + \frac{xy^3}{6} + \frac{y^4}{16}$.] 3. [$x^2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) + 3$, $x^2 + \frac{1}{x^2} + 2\left(x - \frac{1}{x}\right) - 1$, $\frac{x^4}{a^4} - \frac{2x^3y}{a^3b} + \frac{3x^2y^2}{a^2b^2} - \frac{2xy^3}{ab^3} + \frac{y^4}{b^4}$.] 4. [$a^3 + 3a^2b + 3ab^2 + b^3$, $a^3 + 6a^2b + 12ab^2 + 8b^3$, $\frac{a^3}{8} + \frac{a^2b}{4} + \frac{ab^2}{6} + \frac{b^3}{27}$.]

5. $[x^6+3ax^5+6a^2x^4+7a^3x^3+6a^4x^2+3a^5x+a^6$, $1-\frac{3x}{2}+\frac{3x^2}{2}-\frac{7x^3}{8}+\frac{3x^4}{8}-\frac{3x^5}{32}+\frac{x^6}{64}.]$ 6. $[1+6x+21x^2+44x^3+63x^4+54^5+27x^6$, $x^{\frac{3}{2}}-3xy^{\frac{1}{3}}+3x^{\frac{1}{2}}y^{\frac{2}{3}}-y$, $x-6x^{\frac{2}{3}}y^{\frac{1}{3}}+12x^{\frac{1}{3}}y^{\frac{2}{3}}-8y.]$
7. $[x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)$, $x^3+\frac{1}{x^3}-3\left(x^2+\frac{1}{x^2}\right)+6\left(x+\frac{1}{x}\right)-7$, $x-x^{-1}-3(x^{\frac{2}{3}}-x^{-\frac{2}{3}})-5.]$ 8. $[3a^2bc$, $8a^3b^2c^3$, $24a^mb^nc.^{2n}.]$ 9. $[4a-5b$, $4a^2-6a+8.]$ 10. $[a-b+c.]$
11. $[x^3-3x^2y+3xy^2-y^3.]$ 12. $[x^2-x+\frac{1}{4}.]$ 13. $[\frac{x^2}{2}+ax+\frac{a^2}{3}.]$
14. $[x+1-\frac{1}{x}.]$ 15. $[\frac{2x}{7y}-5+\frac{3y}{4x}.]$ 16. $[2x^2-2x+\frac{1}{2}.]$
17. $[\frac{x}{2}-\frac{b}{4}+\frac{a}{6}.]$ 18. $[7x^2-\frac{x}{5}+3.]$ 19. $[\frac{x}{y}+\frac{y}{x}-\frac{1}{\sqrt{2}}.]$ 20. $[a+\frac{x}{2a}-\frac{x^2}{8a^3}+$ &c.] 21. $[x+\frac{x}{2a}+\frac{3x^2}{8a^3}+$&c.]
22. $[1+\frac{x}{a}+\frac{x^2}{2a^2}+$ &c.] 23. $[x^2-a^4.]$ 24. $[x^2-2x+1.]$
25. $[\frac{a}{2}+\frac{2}{3a^2}.]$ 26. $[\frac{x}{y^2}-\frac{y^2}{x}.]$ 27. $[x-4+\frac{2}{x}.]$
28. $[\frac{ac}{b}-\frac{b}{cx}.]$ 29. $[e^x-e^{-x}$, $\frac{x^2}{a^2}-\frac{a^2}{x^2}.]$ 30. $[x-x^{-1}$, $x-1-\frac{1}{x}.]$ 31. $[2x-3y.]$ 32. $[x^2-\frac{3a^2x}{4b}.]$ 33. $[a^2+b^2.]$
34. $[\frac{7}{a^2b^4}+\frac{9}{a^3b^3}+\frac{3}{a^4b^2}.]$ 35. $[x-x^{-1}.]$ 36. $[2x-1.]$

V.

No. 1. [*Ans.* $\frac{31b}{48a}$.] 2. $[\frac{13}{5x}.]$ 3. $[\frac{19}{30a}.]$ 4. [0.]
5. [0.] 6. [1.] 7. [0.] 8. $[\frac{12xy}{9x^2-4y^2}.]$ 9. [1.]
10. [1.] 11. [0.] 12. [0.] 13. [0.] 14. [0.]

15. $[\frac{y}{x+y}.]$ 16. $[\frac{x^4}{(1-x)^3}.]$ 17. $[\frac{1}{1-x^4}.]$ 18. $[\frac{1+x+x^2}{(1-x^4)(1-x)}.]$

19. [0.] 20. [0.] 21. [0.] 22. [0.] 23. [0.] 24. [0.]

25. [0.] 26. [0.] 27. $[\frac{1}{2y^2-1}.]$ 28. [0.] 29. [1.]

30. $[rs+(rt+qs)x+tqx^2.]$ 31. $[\frac{a-bx}{a+bx}.]$ 32. [0.] 33. [0.]

34. $[\frac{18}{x^2-9}.]$ 35. $[\frac{x^2}{(x-1)(x-2)(x-3)}.]$ 36. $[x-y.]$

37. $[\frac{3ab(3a^2+b^2)}{a^4-b^4}.]$ 38. $[\frac{a^3+ab^2+b^3}{(a+b)^3}.]$ 39. [1.]

40. $[\frac{1-x^2+2x}{(3-x)^2}.]$ 41. $[\frac{2(x-a)}{a+x}.]$ 42. $[\frac{1}{(x^3+1)(x^2+1)}.]$

43. $[\frac{x^2-4x+3}{(x+1)(x-2)(x+3)}.]$ 44. $[\frac{2+x+3x^2}{2(1-x^4)}.]$

45. $[\frac{x^2}{(x+1)(x+2)(x+3)}.]$ 46. $[\frac{6(1-2x^2)}{(1-x^2)(1-4x^2)}.]$

47. $[\frac{1-5x}{(1-x)(1+x)^3}.]$ 48. $[\frac{1}{x^2(x-1)(x+4)}.]$

49. $[\frac{3+5x+3x^2}{(1+x)^2(1+2x)^2}.]$ 50. $[\frac{1}{x^3(1-x)(1-x^2)}.]$

51. $[\frac{2x^5-x^2}{(x^2+1)^2(x^2+x+1)^2}.]$ 52. $[\frac{x+1}{x^3(x^2+1)^2}.]$

53. $[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right).]$ 54. $[\frac{y^2+4}{(y-2)(y+3)(y-1)^2}.]$

55. $[x^{2n}+2.]$ 56. $[\frac{2}{x}.]$ 57. $[\frac{3x^2-2x+1}{(x^2-1)(x^4+x^2+1)}.]$

58. $[\frac{7x+8}{(x+1)^2(x^2+x+1)}.]$ 59. $[\frac{x^3}{(x-a)^2(x^3+a^3)}.]$ 60. [1.]

61. $[d.]$ 62. $[2(a+b+c).]$ 63. [0.] 64. $[\frac{4}{ab^2}\times\frac{b+1}{2b+1}.]$

65. $[a.]$ 66. $[\frac{bc+1}{abc+a+c}, \frac{4}{3x}.]$ 67. [0.]

68. $[\frac{1}{(x+a)(x+b)(x+c)}.]$ 69. $[\frac{x}{(x-a)(x-b)(x-c)}.]$

70. $[\frac{x^2}{(x-a)\ (x-b)\ (x-c)}.]$ 71. $[\frac{x^2+x+1}{(x-a)\ (x-b)\ (x-c)}.]$

72. $[\frac{x^2+mx+n}{(x-a)\ (x-b)\ (x-c)}.]$ 73. $[\frac{a^2+x^2}{a^2-x^2}.]$ 74. $[\frac{x+2}{x+3}.]$

75. $[x^2-3x+2.]$ 76. $[x-5.]$ 77. $[\frac{1}{x+1}.]$ 78. $[1.]$ 79. $[1.]$

80. $[1.]$ 81. $[1.]$ 82. $[1.]$ 83. $[1.]$ 84. $[1.]$ 85. $[\frac{x^2}{3y}.]$

86. $[1.]$ 87. $[\frac{a}{b}-\frac{c}{d}.]$ 88. $[\frac{3a^2}{x^2}-\frac{2ab}{5xy}+\frac{b^2}{y^2}.]$

89. $[a^3+\frac{1}{a^3}+a+\frac{1}{a}.]$

V.*

No. 1. [*Ans.* $a^{\frac{2}{3}}bc^{\frac{1}{3}}$; $a^{\frac{7}{12}}b^{\frac{5}{6}}c^{\frac{3}{4}}$; $ab-cx$.] 2. $[9a^{\frac{2}{3}}b^{\frac{4}{3}}-25a^{\frac{4}{3}}b^{\frac{8}{3}}.]$ 3. $[a-y.]$ 4. $[a+b.]$ 5. $[x^{\frac{3}{2}}+x+x^{\frac{1}{2}}.]$ 6. $[a^{\frac{1}{2}}-b^{\frac{1}{2}}.]$ 7. $[9x+14x^{\frac{2}{3}}+25x^{\frac{1}{3}}.]$ 8. $[3(x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}.]$ 9. $[a-b.]$ 10. $[x-mp.]$ 11. $[a^{\frac{1}{2}}-2a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}.]$ 12. $[1+y^{\frac{1}{12}}+y^{\frac{1}{6}}+y^{\frac{1}{4}}+y^{\frac{1}{3}}+y^{\frac{5}{12}}+y^{\frac{1}{2}}+y^{\frac{7}{12}}+y^{\frac{2}{3}}.]$ 13. $[3a^{\frac{1}{2}}b^{\frac{2}{3}}-5a^{\frac{1}{3}}b^{\frac{2}{3}}.]$ 14. $[x^{\frac{3}{8}}+x^{\frac{1}{8}}-x^{-\frac{1}{8}}-x^{-\frac{3}{8}}.]$ 15. $[a^{\frac{3}{8}}+a^{\frac{1}{4}}b^{\frac{1}{8}}+a^{\frac{1}{8}}b^{\frac{1}{4}}+b^{\frac{3}{8}}.]$ 16. $[x^4-2+x^{-4}.]$

17. $[x^{\frac{3}{8}}-x^{-\frac{3}{8}}.]$ 18. $[a^{\frac{1}{6}}-a^{\frac{1}{12}}x^{\frac{1}{12}}+x^{\frac{1}{6}}.]$ 19. $[x^{\frac{1}{6}}+\frac{x^{\frac{1}{12}}}{2}+\frac{1}{4}.]$

20. $[x^{\frac{1}{6}}-x^{-\frac{1}{6}}+1.]$

VI.

No. 3. [*Ans.* 720.] 4. $[-360.]$ 5. $[11\sqrt{3}.]$ 6. $[13\sqrt[3]{5}.]$ 7. $[a^2b^{\frac{4}{3}}.]$ 8. $[15\sqrt{2}.]$ 9. $[12\sqrt{3}+14\sqrt{2}.]$ 10. $[2\sqrt[3]{4}+7\sqrt{7}+8\sqrt{3}.]$ 11. $[21(\sqrt{2}+\sqrt{3}).]$ 12. $[8\sqrt{2}.]$ 13. $[\frac{25\sqrt{3}}{2}.]$

14. $[18\sqrt{2}.]$ 15. $[48\sqrt{3}.]$ 16. $[70\sqrt[3]{12}.]$ 17. $[\frac{5}{3}\sqrt{\frac{5}{2}}.]$

18. $[\sqrt[3]{\frac{2}{3}}.]$ 19. $[17\frac{22}{25}.]$ 20. $[\frac{22}{14}\sqrt{\frac{5}{11}}.]$ 21. $[\sqrt{ay^{-1}}.]$

22. $[2a^2\sqrt[3]{b}.]$ 23. $[\sqrt[6]{x^{-1}y^2}.]$ 24. $[48\sqrt[3]{4}.]$ 25. $[6a\sqrt{b+x}.]$

26. $[(4a^2-2ax)\sqrt{5x}.]$ 27. $[\frac{1}{12}\sqrt[3]{18},\ \frac{1}{3\sqrt{6}}.]$ 28. $[5a\sqrt{b}.]$

29. $[a^{-\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}.]$ 30. [37; 28.] 31. $[-20(1+\sqrt{-5});$ $105.]$ 32. $[-a^{\frac{1}{2}}b^{\frac{1}{2}};\ a^{\frac{1}{4}}b^{\frac{1}{4}}\sqrt{-1}.]$ 33. $[x^4+2x^3-8x^2-6x-1.]$

34. $[(x^2-y^2)^{\frac{m+n}{mn}}.]$ 35. [2.] 36. $[12ax\sqrt{3x}.]$ 37. $[a^2b^{\frac{4}{3}}.]$

38. $[\frac{a\sqrt{ab}}{2c}.]$ 39. $[\frac{1}{10};\ \sqrt{1\mp\frac{2}{\sqrt{5}}}.]$ 40. $[\sqrt{5\mp 2\sqrt{5}};$ $\sqrt[4]{10}+\sqrt[4]{5}.]$ 41. $[(a^2-b^2)^{\frac{m+n}{mn}}.]$ 42. $[x^{-\frac{1}{3}}-x^{\frac{2}{3}}.]$ 43. $[a^{\frac{1}{3}}+y^{\frac{1}{3}}.]$

44. $[y^{\frac{1}{4}}x^{-\frac{7}{20}}.]$ 45. $[ax^2y^6.]$ 46. $[\frac{a\sqrt{b}}{\sqrt{b}\mp\sqrt{c}}.]$

47. $[x^{\frac{m}{2}}-\frac{1}{2}b^{\frac{1}{n}}x^{\frac{2}{p}}.]$ 48. $[\frac{x-y\sqrt{2}}{x\sqrt{2}-y}.]$ 49. $[b(3a+5b).]$

50. $[\frac{1}{1-x^2}.]$ 51. $[\frac{1+x^2}{(1-x^2)^{\frac{3}{2}}}.]$ 52. $[\sqrt{a^2-x^2}+\sqrt{a^2+x^2}.]$

53. $[2x^2.]$ 54. $[2x.]$ 55. $[a+\sqrt{b}.]$ 56. $[\frac{x^2+3y^2}{x^2-3y^2}.]$

57. $[a^{-\frac{1}{xy}}-a^{-\frac{y}{2r}}b^{\frac{1}{2r}}.]$ 59. $[\frac{17+7\sqrt{5}}{11};\ 16-3\sqrt{15}.]$

60. $[8\sqrt{3}-7\sqrt{2};\ \sqrt{3}+1+\sqrt{2}.]$ 61. $[\sqrt{2}.]$ 62. $[\frac{1}{7}.]$

63. [9.] 64. $[\frac{2(a^2-b)}{a^2+b}]$ 65. $[\frac{x^{\frac{2}{3}}+y}{x^{\frac{2}{3}}-y}]$ 66. $[\frac{a^{\frac{1}{2}}}{(a-b)(x-a)}.]$

67. $[\frac{a+b+c}{\sqrt{(a+b-c)(a+c-b)(b+c-a)}}.]$ 68. $[\frac{2a}{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2}.]$

69. $[\frac{3x^2}{x^3-1}.]$ 70. $[\frac{3+2\sqrt{2}+\sqrt{6}+\sqrt{3}}{2\sqrt{2}}.]$ 71. $[\frac{\sqrt{6}}{3}.]$

73. $[5+\sqrt{2};\ 1+\sqrt{7}.]$ 74. $[\cdot 2+\cdot 1\sqrt{-1};\ 5-2\sqrt{3}.]$

75. $[5-\sqrt{3};\ 1+\sqrt{-3}.]$ 76. $[5-2\sqrt{-1};\ 2-\sqrt{-5}.]$

77. $[\frac{1+\sqrt{3}}{\sqrt{2}};\ 3-3\sqrt{-1}.]$ 78. $[2^{-\frac{1}{2}}(\sqrt{7}-1);\ 2\pm 2\sqrt{3}.]$

79. $[2+\sqrt{3};\ 3^{-\frac{1}{2}}(1-\sqrt{2}).]$ 80. $[\frac{1+\sqrt{3}}{\sqrt[4]{2}};\ a(\sqrt{2}+\sqrt{-2}).]$

81. [$x+\sqrt{a^2-x^2}$; $1-x+\sqrt{1+2x-x^2}$.] 82. [$\sqrt{\frac{1+m}{2}}+\sqrt{\frac{1-m}{2}}$; $x-\sqrt{xy-x^2}$.] 83. [$1+\sqrt{5}$; $1+\sqrt{7}$.] 84. [$\sqrt{2}+\sqrt{3}$; $\frac{1}{2}(\sqrt{7}+\sqrt{3})$.] 85. [$1-2\sqrt{-1}$; $1+\sqrt{3}+\sqrt{5}$.] 86. [$x^{\frac{1}{2}}-x^{\frac{1}{3}}+x^{\frac{2}{4}}$.]
90. [$\sqrt[3]{3}$ the greatest, and $\sqrt[5]{5}$ the least.]

VII.

No. 1. [*Ans.* 8.] 2. [$\frac{2}{3}$.] 3. [1.] 4. [9.] 5. [3.] 6. [12.] 7. [4.] 8. [2.] 9. [7.] 10. [$-\frac{13}{2}$.] 11. [$\frac{5}{9}$.] 12. [12.] 13. [$1\frac{13}{87}$.] 14. [9.] 15. [$15\frac{1}{18}$.] 16. [$4\frac{7}{9}$.] 17. [$11\frac{4}{5}$.] 18. [17.] 19. [5.] 20. [$7\frac{1}{18}$.] 21. [$\frac{3}{5}$.] 22. [$\frac{67}{83}$.] 23. [7.] 24. [17.] 25. [4.] 26. [14.] 27. [11.] 28. [3.] 29. [7.] 30. [$\frac{2}{3}$.] 31. [17.] 32. [$1\frac{2}{5}$.] 33. [21.] 34. [18.] 35. [7.] 36. [6.] 37. [8.] 38. [$\frac{5}{2}$.] 39. [1.] 40. [4.] 41. [8.] 42. [10.] 43. [3.] 44. [$1\frac{2}{3}$.] 45. [-2.] 46. [$2\frac{1}{3}$.] 47. [$-19\frac{2}{3}$.] 48. [$1\frac{9}{11}$.] 49. [$3\frac{3}{8}$.] 50. [$\frac{4}{13}$.] 51. [4.] 52. [$\frac{4}{7}$.] 53. [$-4\frac{1}{2}$.] 54. [3.] 55. [5.] 56. [8.] 57. [4.] 58. [2.] 59. [$1\frac{1}{3}$.] 60. [5.] 61. [8.] 62. [6.] 63. [2.] 64. [30.] 65. [2.] 66. [$\frac{70ab-3ac}{320c}$.] 67. [$\frac{4ab^2-10a}{4a-3b}$.] 68. [$\frac{a^2+b^2-c^2}{2b}$.] 69. [$\frac{a^2}{a-b}$.] 70. [$\frac{5b^2}{16c}$.] 71. [$7a$.] 72. [$12a$.] 73. [$-\frac{3a}{4}$.] 74. [$\frac{5a+16}{7}$.] 75. [$\frac{-63a}{79}$.] 76. [$4a$.] 77. [$8a$.] 78. [$\frac{3}{2a^2}$.] 79. [$\frac{a}{2b}$.] 80. [$\frac{2a+c}{3}$.] 81. [$\frac{b(b+c)-(a^2+c^2)}{(a-c)^2-b(b-c)}$.] 82. [$\frac{b(a+c-b)}{a}$.] 83. [$\frac{ac+bc}{a}$.] 84. [$a^3+a^2b+b$.] 85. [$-\frac{ab+bc+ac}{a+b+c}$.] 86. [$\frac{ab}{a-b}$.] 87. [$\frac{ab}{2(a-b)}$.] 88. [$a^{\frac{1}{2}}b^{\frac{1}{2}}-(a+b)$.] 89. [$-11\frac{3}{4}$.] 90. [9.] 91. [4.] 92. [9.] 93. [25.] 94. [$-2\frac{2}{3}$.] 95. [2.] 96. [$\frac{1}{3}$.] 97. [16] 98. [4.]

99. [5, $\frac{2}{3}$.] 100. [$\frac{\sqrt{a}}{\sqrt{a}+2}$.] 101. [$\frac{(a-b)^2}{2b}$.] 102. [0.]

103. [$\frac{a}{2}\sqrt{\frac{a^2-4}{a^2-1}}$.] 104. [$\frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}$.] 105. [$\frac{a(n^2-1)}{\sqrt{2n^2-1}}$.]

106. [$\frac{(a^2-2a+2)^2}{4(a-1)^2}$.] 107. [$\frac{8ab}{\sqrt{1-16a^2}}$.] 108. [$\frac{(a^2-1)^2}{a^4+2a^2-3}$.]

109. [$\frac{a\sqrt{3b}}{\sqrt{2\sqrt{a^2-ab}}}$.] 110. [$\frac{5a}{4}$.] 111. [$\frac{9n}{16}$.] 112. [30.]

113. [$\frac{1}{a}\sqrt{\frac{2a}{b}-1}$.] 114. [$\frac{\sqrt{3}}{2}$.] 115. [$\frac{a^2}{2a+b}$.]

116. [$\frac{a(\sqrt{c}\mp 2\sqrt{b})}{\sqrt{c}}$.] 117. [$\frac{a-b}{2\sqrt{ab}}$.] 118. [$\frac{c^2-a^2}{a}$.]

119. [1.] 120. [$\frac{a^2+2ab}{2b}$.] 121. [$\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-1}$.]

122. [$\frac{1}{6}$.] 123. [$\frac{a}{2}\left(1\pm\frac{1}{\sqrt{2}}\right)$.] 124. [$a+8$.] 125. [$\frac{3a}{4}$.]

126. [$\frac{(a^2-2ab+2b^2)^2}{4b^2(a-b)^2}$.] 127. [$-\frac{3a}{7}$.] 128. [$\frac{1}{a}\left(b-\frac{nc}{n-1}\right)^2$.]

129. [$\frac{2}{m}\sqrt{\frac{m^2-1}{m^2-4}}$.] 130. [1.] 131. [$\sqrt{a^2-\left(\frac{b^2}{3}-\frac{2a}{3b}\right)^3}$.]

132. [$-\frac{5a}{2}$.] 133. [$\sqrt{\frac{a-2}{a+4}}$.] 134. [$\frac{a-b}{2\sqrt{ab}}$.] 135. [$n\sqrt{a-\frac{n^2}{4}}$.]

136. [$\frac{\sqrt{3}}{2}$.] 137. [$\sqrt{1+a^2}\times\frac{1-a^2+a\sqrt{1+a^2}}{2a}$.] 138. [$\frac{1}{12}\times\frac{3a-1}{a+1}$.] 139. [$\sqrt{(2-a)^2-1}$.] 140. [$\frac{(a+2)^2}{4a(a+1)}$.] 141. [$-a$.]

142. [$\frac{1}{2}\left(\sqrt[3]{a}+\frac{1}{\sqrt[3]{a}}\right)-1$.] 143. [$\frac{a}{1+2a^{\frac{3}{2}}}$.] 144. [$a.\frac{(1\pm\sqrt{b})^2}{1+b}$.]

145. [$4a$.] 146. [$\pm\frac{a\sqrt{3}}{2}$.] 147. [$\frac{4}{a}\sqrt{1-\left(\frac{2}{a}\right)^2}$.]

148. [$\sqrt[4]{\frac{1}{2}+\frac{1}{\sqrt{2}}}$.] 149 [$a\left\{1-\left(\frac{2\sqrt{b}}{b+1}\right)^4\right\}$.]

150. $[\frac{2a\sqrt{b}}{1+b^2}\sqrt{1-b^2}.]$ 151. $[\pm\sqrt{2}, \pm 1$, or $\pm\sqrt{4(a-1)^2+1}.]$

152. $[\sqrt[n]{\frac{2ac}{b(c^2+1)}}.]$ 153. $[\frac{2a}{3}\sqrt[4]{5}.]$ 154. $[\frac{4}{9}.]$

155. $[\pm 2\sqrt{(1-a)\left(1-\frac{a}{3}\right)}.]$ 156. $[\frac{1}{a}\sqrt{\frac{2a}{b}-1}.]$

157. $[\frac{1}{a}\pm\sqrt{\left(\frac{1}{a}\right)^2-1}.]$ 158. $[(\sqrt{a}+\sqrt{b})^2.]$

159. $[\frac{3a-1}{\sqrt{(1-a)(9a-1)}}.]$ 160. $[\frac{a}{2b}(1-b)^2.]$ 161. [3.]

162. $[a-\frac{a}{\sqrt{1+(b-1)^2}}.]$ 163. $[\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^{\frac{1}{2}}.]$

164. $[\pm 2\sqrt{a}+a^{-1}.]$ 165. $[\frac{(a+b)^2}{4ab}\left(1+\frac{8ab}{(a-b)^2}\mp\sqrt{1+\frac{8ab}{(a-b)^2}}\right).]$ 166. [8.] 167. $[\sqrt{1-a^2}\times\frac{1\pm\sqrt{1-a^2}}{a^2+\sqrt{2a^2-1}}.]$

VIII.

No. 1. [*Ans.* ± 3.] 2. [± 4.] 3. [± 1.] 4. [± 3.] 5. [$\pm\frac{1}{2}$.]
6. [± 5.] 7. [10, 2.] 8. [8, 2.] 9. [8, -40.] 10. [3, -1.]
11. [-1, -12] 12. [17, -4.] 13. [-5, -20.] 14. [1, -20.]
15. [6, $-\frac{17}{3}$.] 16. [$\frac{3}{2}$, $\frac{3}{10}$.] 17. [6, $-\frac{21}{2}$.] 18. [3, $-\frac{9}{4}$.]
19. [2, $\sqrt[3]{-6}$.] 20. [± 3, $\pm\sqrt{-\frac{34}{5}}$.] 21. [2, $\sqrt[3]{-4}$.]
22. [3, $\sqrt[3]{-41}$.] 23. $\sqrt{2}$, $\sqrt{-8}$.] 24. [$\sqrt{3}$, $\sqrt{2}$.]
25. [4, or $-5\frac{6}{11}$.] 26. [6, $3\frac{1}{13}$.] 27. [8, $-\frac{4}{27}$.] 28. [8, $13\frac{22}{31}$.]
29. [3, $-3\frac{23}{33}$.] 30. [9, 1.] 31. [5, $6\frac{9}{10}$.] 32. [7, $-1\frac{3}{7}$.]
33. [3, $-3\frac{1}{2}$.] 34. [4, $\sqrt{-190}$.] 35. [4, $-1\frac{2}{3}$.] 36. [6, -4.]
37. [3, $-\frac{4}{3}$.] 38. [2, $\frac{1}{18}$.] 39. [$\frac{1}{2}$, $-\frac{1}{3}$.] 40. [1, or -4.]
41. [3, -3.] 42. [9, $-\frac{25}{31}$.] 43. [± 3, $\pm\sqrt{\frac{39}{11}}$]
44. [0, ± 9.] 45. [1, $\frac{-3\pm\sqrt{5}}{2}$.] 46. [2, 1, $\frac{-9\pm\sqrt{73}}{2}$.]
47. [$\pm\sqrt{\frac{29}{3}}$, $\pm\sqrt{\frac{-22}{3}}$.] 48. [4, 2, $\frac{-7\pm\sqrt{17}}{2}$.]

49. $[\pm 3, \pm\sqrt{-15}.]$ 50. $[9, 5 \pm (-5)^{\frac{2}{3}}.]$ 51. $[a, b.]$

52. $[1, -\frac{a}{2b}.]$ 53. $[\frac{d}{c}, -\frac{b}{a}.]$ 54. $[a+\frac{b}{c}, a-\frac{c}{b}.]$

55. $[\frac{a \pm b}{ab}.]$ 56. $[a+b, -c.]$ 57. $[\pm\frac{ab^{\frac{1}{2}}}{a^{\frac{1}{2}} \pm b^{\frac{1}{2}}}.]$

58. $[\frac{a^2}{b}, -\frac{2a^2}{b}.]$ 59. $[\frac{a \pm \sqrt{a^2+b^2}}{3a^2b^2}.]$ 60. $[\frac{3a}{4}, \text{ or } \frac{a}{2}.]$

61. $[\frac{a(-1 \pm \sqrt{2})}{2}.]$ 62. $[\frac{n}{q}, -\frac{p}{m}.]$ 63. $[-\frac{3a}{4}, -\frac{3b}{2}.]$

64. $[12b, -\frac{3a}{2}.]$ 65. $[\frac{b \pm a}{3}.]$ 66. $[\frac{b \pm a}{5}.]$ 67. $[\frac{b \pm a}{2}.]$

68. $[\frac{b}{a}, -\frac{b}{6a}.]$ 69. $[3a, 2b.]$ 70. $[\frac{n}{m}, -\frac{m}{n}.]$ 71. $[\frac{m \pm n}{2}.]$

72. $[\frac{3a \pm 2b}{2}.]$ 73. $[\frac{1}{2p}(-q \pm \sqrt{q^2-4p^2}).]$ 74. $[\mp\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2-4q}.]$

75 $[1, \frac{2b}{a-b}.]$ 76. $[a, \frac{b}{na}.]$ 77. $[\frac{2c}{a+b}, \frac{-c}{a+b}.]$ 78. $[\frac{a \pm b}{2}.]$

79. $[\frac{b}{c}(-a^2 \pm \sqrt{c^2+a^4}).]$ 80. $[\frac{a}{2}(-3 \pm \sqrt{3}).]$ 81. $[\frac{1}{2(b+c)} \times (d \pm \sqrt{4ab+d^2}).]$ 82. $[\pm a\sqrt{\frac{3}{2}}.]$ 83. $[\pm\sqrt{\frac{n}{n-2}}.]$

84. $[a+b, c-b.]$ 85. $[\pm\frac{a\sqrt{6}}{\sqrt{5}}.]$ 86. $[\frac{a+\sqrt{ab}+b}{a-\sqrt{ab}+b}, \frac{a-\sqrt{ab}+b}{a+\sqrt{ab}+b}.]$

87. $[\frac{2a+3b}{2a-3b}, \frac{3a-2b}{3a+2b}.]$ 88. $[1, \frac{2b}{a-b}.]$ 89. $[a, \frac{b^2}{n^2a}.]$

90. $[\frac{ac}{b}, \frac{bc}{a}.]$ 91. $[-\frac{(\sqrt{a} \mp \sqrt{b})^2}{\sqrt{ab}}.]$ 92. $[25, 16.]$ 93. $[9, 4.]$

94. $[5, -\frac{179}{28}.]$ 95. $[-1, -8.]$ 96. $[\pm 1, 2\sqrt{-1}, -\frac{3 \pm 3\sqrt{5}}{2}]$ 97. $[2, \text{ or } 3\frac{1}{2}.]$ 98. $[\pm 5, \pm\sqrt{38}.]$ 99. $[-\frac{1}{3}, \text{ or } \frac{4}{3}.]$

100. $[3, \text{ or } -\frac{5}{9}.]$ 101. $[2, \text{ or } -\frac{1}{2}.]$ 102. $[\pm 2a\sqrt{-2}, \pm a\sqrt{-15}.]$

103. $[a, \frac{a \pm \sqrt{5a^2-8an}}{2}.]$ 104. $[16, 9.]$ 105. $[9, \text{ or } 4.]$

106. $[\frac{-n}{a-9n}, \frac{-n}{a-n}.]$ 107. $[4, -1, \frac{3 \pm \sqrt{5}}{2}.]$ 108. $[3, -4, -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{259}{3}}.]$ 109. $[1, \frac{1}{5}.]$ 110. $[2$, or $\frac{1}{4}, \frac{9 \pm \sqrt{-31}}{8}.]$

111. $[\pm 5, \pm 3\sqrt{2}.]$ 112. $[\pm\sqrt{6}$, or $\pm\sqrt{3}.]$ 113. $[\pm\frac{4}{5}, \pm\frac{\sqrt{21}}{5}.]$

114. $[\pm \frac{1}{2}.]$ 115. [5, or 3.] 116. $[\pm\sqrt{2ab-b^2}$, or $0.]$

117. $[\pm a\frac{\sqrt{3}}{2}.]$ 118. $[\pm a$, or $\pm\frac{a}{\sqrt{2}}.]$ 119. $[\frac{a}{(1\pm\sqrt{b})^2}.]$

120. $[\frac{2}{m}\sqrt{\frac{m^2-1}{m^2-4}}.]$ 121. $[\frac{4n(1-n^2)}{(1+n^2)^2}]$ 122. $[\pm 3a\sqrt{-1}.]$

123. $[8, -\frac{8}{9}.]$ 124. $[0, \pm\sqrt{-3}.]$ 125. $[\pm\frac{1}{2}\sqrt{3}.]$

126. $[\pm\frac{b}{\sqrt{2ab-1}}.]$ 127. $[\pm\frac{(b^2+1)a}{2b}.]$ 128. $[\left\{\frac{2am}{b(m^2+1)}\right\}^{\frac{1}{n}}.]$

129. $[\pm\sqrt{2ab-b^2}.]$ 130. $[12, -3.]$ 131. $[9\pm\frac{27}{\sqrt{7}}.]$

132. $[\pm 5.]$ 133. $[4, 1, \frac{-3\pm\sqrt{-7}}{2}.]$ 134. $[\pm 5, \pm 4\sqrt{2}.]$

135. $[4, (-5)^{\frac{2}{3}}.]$ 136. $[4, \frac{1}{2}(3\mp 2\sqrt{5}).]$ 137. $[1, \frac{1\pm\sqrt{-3}}{2}.]$

138. $[1, \frac{1\pm\sqrt{5}}{2}.]$ 139. $[1, \frac{5\pm\sqrt{5}}{2}.]$ 140. $[4, -8, \frac{-2\pm 2\sqrt{-71}}{9}.]$

141. $[1\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}}.]$ 142. $[\frac{1\pm\sqrt{-15}}{4}.]$ 143. $[\frac{1}{2}(1\pm\sqrt{3\pm 2\sqrt{5}}).]$ 144. $[4$, or $-3, \frac{1\pm\sqrt{-43}}{2}.]$ 145. $[\pm 1, \pm\sqrt{-1}, \pm\frac{1\pm\sqrt{-3}}{2}, \pm\sqrt{\frac{1\pm\sqrt{-3}}{2}}.]$ 146. $[1, \frac{-1\pm\sqrt{-7}}{4}.]$

147. $[-1, \frac{1-p\pm\sqrt{p^2-2p-3}}{2}.]$ 148. $[-1, \frac{1\pm\sqrt{4p-3}}{2}.]$

149. $[5, -1, 2\pm\sqrt{5}.]$ 150. $[\frac{-5\pm\sqrt{21}}{2}, \pm\sqrt{-1}.]$

151. $[\pm\sqrt{-1},\ 2,\ \frac{1}{2}.]$ 152. $[1, \text{or } \frac{1\pm\sqrt{-15}}{4}.]$ 153. $[1,$
or $\frac{1}{2}(3\pm\sqrt{5}.)]$ 154. $[\frac{8a}{5},\ \frac{2a}{7}\left(-1\pm\frac{2\sqrt{-1}}{\sqrt{3}}\right)]$ 155. $[\frac{1}{2}(b\pm$
$\sqrt{b^2-2ab}.]$ 156. $[-a,\ a(3-4a^2).]$ 157. $[\pm\sqrt{\frac{1\pm\sqrt{5}}{2}}.]$
158. $[a\,\frac{(3\pm\sqrt{5})^5-2^5}{(3\pm\sqrt{5})^5+2^5}.]$ 159. $[\frac{a}{2},\ -a.]$ 160. $[\frac{2a}{3},\ -2a.]$
161. $[\frac{a}{8}(-3\pm\sqrt{\frac{1051}{3}}.]$ 162. $[\frac{49\pm\sqrt{97}}{8}.]$ 163. $[\frac{a}{7}\times$
$(9\pm4\sqrt{2}),\ \frac{a}{5}(4\pm3\sqrt{-1}).]$ 164. $[\pm\sqrt{\frac{2}{\sqrt{3}}}.]$ 165. $[93,\ 7.]$
166. $[4\frac{1}{2},\ 3.]$ 167. $[5,\ -\frac{22}{15}.]$ 168. $[\frac{5}{2},\ \frac{5}{8}.]$ 169. $[2a^2\times$
$\left(1\pm\sqrt{1+\frac{1}{a}}\right).]$ 170. $[\pm2,\ \pm\sqrt{-3}.]$ 171. $[4,\ \frac{49}{16}.]$

172. $[\left(\frac{a\pm b}{a\mp b}\right)^{\frac{2pq}{q-p}}.]$ 173. $[\left\{\frac{a^{\frac{r}{s}}+1}{a^{\frac{r}{s}}-1}\right\}^{\frac{1}{m^2+2mn+n^2}}.]$

174. $[\left(\frac{\sqrt{a}\pm\sqrt{b}}{\sqrt{ab}}\right)^{\frac{4mn}{m-n}}.]$ 175. $[\left(\frac{a^{2m}\pm1}{a^{2m}\mp1}\right)^{2}.]$ 176. $[\sqrt[m]{-1},$
$(\frac{1}{2}\pm\frac{1}{2}\sqrt{4a+1})^{\frac{1}{m}}.]$ 177. $[\frac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}.]$ 178. $[(\frac{1}{2}+\frac{1}{a}\pm$
$\sqrt{\frac{1}{4}-\frac{1}{a}})^{\frac{1}{2}}.]$ 179. $[\frac{28a}{27}.]$ 180. $[0,\ 8.]$ 181. $[3,\ -\frac{3}{4},$
$\frac{81\pm9\sqrt{97}}{8}.]$ 182. $[\frac{63a}{85},\ 0.]$ 183. $[\frac{ab^{\frac{2}{3}}}{(a^{\frac{2}{3n}}+b^{\frac{2}{3n}})^n}.]$

184. $[\frac{1}{2}(3\pm\sqrt{13}),\ 9\pm4\sqrt{7}.]$ 185. $[a,\ \frac{a}{12}(7\pm\sqrt{-47}).]$
186. $[\frac{1}{4}(3\pm\sqrt{-7}),\ \frac{1}{4}(-1\pm\sqrt{-15}).]$ 187. $[\frac{1+2a\pm\sqrt{12a-3}}{2(1-a)}.]$
188. $[\pm\left(\frac{m-4}{2m}\pm\frac{\sqrt{16-3m^2}}{2m}\right)^{\frac{1}{2}}.]$ 189. $[a\sqrt{-1},\ a+m+\sqrt{a^2+2am},$

190. $\left[\left(\frac{a+4\pm 2\sqrt{3a+3}}{a-2}\right)^{\frac{1}{2}}.\right]$ 191. $[1\pm\sqrt{3}\pm\sqrt{3\pm 2\sqrt{3}}.]$

192. $[m\pm\sqrt{m^2-1}$, where $2m=\frac{2a\pm\sqrt{2+2a}}{a-1}.]$ 193. $[m\pm\sqrt{m^2-1}$,

where $m=\frac{1}{2(a-1)}(4a+1\pm\sqrt{4a+5}).]$ 194. $[\sqrt{m\pm\sqrt{m^2-1}}$;

where $2m=\frac{2}{a^2}(-1\pm\sqrt{1-a^4}).]$ 195. $[\frac{1}{a}, \frac{1-3a^2}{a(a^2-3)}.]$

196. $[\frac{(1\pm\sqrt{-3})^5-2^5}{(1\pm\sqrt{-3})^5+2^5}.]$ 197. $[a\left(\frac{3^m-1}{3^m+1}\right)^2$, or $\frac{(-2)^m-1}{(-2)^m+1}a.]$

198. $[\pm\frac{1}{2}\pm\frac{1}{2}\sqrt{-7}.]$ 199. $[\sqrt[3]{\frac{9a^2b-4b^3}{4}}.]$ 200. $[\frac{1}{a}-\sqrt{\frac{1}{a^2}-1}.]$ 201. $[2m(m\pm\sqrt{a})$, $2m=-\frac{a}{2}\pm\frac{a}{2}\sqrt{a^2+4b}.]$

202. $[\pm\sqrt{(n\pm(n-1)\sqrt{1-a^2})^2-a^2}.]$ 203. $[2401, \frac{1}{16}(-7\pm\sqrt{37})^4.]$

204. $[a, -b.]$ 205. $[4, 9, \frac{-13\pm\sqrt{-27}}{2}.]$ 206. $[\pm\sqrt{\left\{a^2-\left(b^2\pm\sqrt{a+\frac{b^4}{2}}\right)^4\right\}}.]$ 207. $[\frac{a}{2}\left(1\pm\frac{1}{\sqrt{2}}\right).]$ 208. $[3, -\frac{3}{2}$, $\frac{1}{8}(81\pm 9\sqrt{57}).]$ 209. $[3, -\frac{3}{4}, \frac{3}{8}(9\pm\sqrt{97}).]$ 210. $[\frac{\pm\sqrt{a+1}-1}{\pm\sqrt{a-1}+1}.]$

211. $[\frac{a}{2}\sqrt[3]{\frac{a}{2b^3}}+\frac{ab}{2}\sqrt[3]{\frac{2}{ab}}-a.]$ 212. $[0, 2\pm 2\sqrt{2}.]$

213. $[a+2\pm\sqrt{8a+3}.]$ 214. $[12\pm\sqrt{114}\pm 2\sqrt{257\pm 24\sqrt{114}}.]$

IX.

No. 1. [*Ans.* $x=3$, $y=4$.] 2. [$x=5$, $y=2$.] 3. [$x=6$, $y=12$.]
4. [$x=480$, $y=672$.] 5. [$x=7$, $y=9$.] 6. [$x=4$, $y=7$.]
7. [$x=12$, $y=15$.] 8. [$x=3$, $y=5$.] 9. [$x=9$, $y=2$.]
10. [$x=\frac{1}{4}$, $y=\frac{1}{5}$.] 11. [$x=3$, $y=2$.] 12. [$x=5$, $y=7$.]
13. [$x=8$, $y=4$.] 14. [$x=99$, $y=15$.] 15. [$x=4$, $y=3$.]

16. [$x=3$, $y=2$.] 17. [$x=21$, $y=20$.] 18. [$x=7$, $y=4$.]

19. [$x=\cdot 02$, $y=2\cdot 9$.] 20. [$x=10$, $y=5$.] 21. [$x=\dfrac{a^2}{a-b}$, $y=\dfrac{-b^2}{a-b}$.] 22. [$x=-\frac{1}{5}$, $y=-\frac{1}{18}$.] 23. [$x=\dfrac{2b^2-5a^2}{3a}$, $y=\dfrac{4a^2-b^2}{3b}$.] 24. [$x=\dfrac{m^2-n^2}{ma-nb}$, $y=\dfrac{m^2-n^2}{mb-na}$.] 25. [$x=\dfrac{ab}{a+b}$, $y=\dfrac{a^2}{a+b}$.] 26. [$x=\dfrac{b'm-bn}{ab'-a'b}$, $y=\dfrac{an-a'm}{ab'-a'b}$.] 27. [$x=\dfrac{(ab+a-b)\,ab}{a^2b^2+a^2-b^2}$, $y=\dfrac{(a-b-ab)\,ab}{a^2b^2+a^2-b^2}$.] 28. [$x=c\,\dfrac{a^2+b^2}{a^2-b^2}$, $y=c\,\dfrac{a^2+b^2}{2ab}$.] 29. [$x=\dfrac{ab}{a-b}$, $y=\dfrac{ab}{a+b}$.] 30. [$x=\dfrac{ab}{nm}\times\dfrac{n^2-m^2}{nb-ma}$, $y=\dfrac{ab}{nm}\times\dfrac{n^2-m^2}{an-bm}$.] 31. [$x=\dfrac{2mb}{a\,(a+b)}$, $y=\dfrac{m}{b}\times\dfrac{a-b}{a+b}$.] 32. [$x=\left(\dfrac{a+b}{a-b}\right)^{\frac{1}{2}}$, $y=\left(\dfrac{a-b}{a+b}\right)^{\frac{1}{2}}$.]

33. [$x=\dfrac{-c}{a+b}$, $y=\dfrac{c}{a+b}$.] 34. [$x=\dfrac{a\,(a+b)}{a-b}$, $y=\dfrac{a\,(a-b)}{a+b}$.]

35. [$x=\dfrac{a+b}{4}$, or $\dfrac{b^2}{4a^2}(a+b)$; $y=\dfrac{a-b}{4}$, or $\dfrac{b^2}{4a^2}(a-b)$.] 36. [$x=\dfrac{a-b}{2}$, $y=\dfrac{a-b}{2}$ also $x=y=a+b$.] 37. [$x=3$, $y=2$.]

38. [$x=\pm 6$; $y=\pm 3$.] 39. [$x=a-b$, or 0; $y=-b$, or $-a$.]

40. [$x=6$, or -4; $y=2$, or -3.] 41. [$x=5$, or $\frac{3}{4}$; $y=3$, or $-\frac{5}{4}$.]

42. [$x=\pm 9$, $y=\pm 5$.] 43. [$x=3$, or $-\frac{29}{7}$; $y=\frac{32}{17}$, or 1.]

44. [$x=7$, or -3; $y=3$, or -7.] 45. [$x=9$, or $-\frac{17}{2}$; $y=-8$, or $\frac{19}{2}$.] 46. [$x=7$, $y=6$.] 47. [$x=8$, or $\frac{6}{7}$; $y=6$, or $-\frac{8}{7}$.]

48. [$x=\pm 6$, $y=\pm 5$.] 49. [$x=2$, or -1; $y=1$, or -2.] 50. [$x=\pm 6$, or $\pm\dfrac{1}{\sqrt{2}}$; $y=\pm 5$, or $\pm\dfrac{-11}{\sqrt{2}}$.] 51. [$x=4$, or 2; $y=2$, or 4.]

52. [$x=3$, or 2; $y=2$, or 3.] 53. [$x=6$, or 3; $y=3$, or 6.]

54. [$x=\pm 5$, $y=\pm 4$.] 55. [$x=6$, $y=5$.] 56. [$x=11$, $y=3$.]

57. [$x=8$, or 2; $y=4$.] 58. [$x=11$, $y=3$.] 59. [$x=5$, $y=2$.]

60. [$x=3$, or 2; $y=2$, or 3.] 61. [$x=3$, $y=2$.] 62. [$x=1$, or $\pm\frac{2}{\sqrt{5}}$; $y=1$, or $\pm\frac{7}{2\sqrt{5}}$.] 63. [$x=1$, or 2; $y=2$, or 1.]

64. [$x=2$, or 1; $y=1$, or 2.] 65. [$x=2, y=1$.] 66. [$x=2$, $y=1$.] 67. [$x=-1$, or $\frac{7}{5}$; $y=1$, or $-\frac{8}{5}$.] 68. [$x=4, y=3$.]

69. [$x=3$, or -2; $y=-2$, or 3.] 70. [$x=\sqrt[4]{\frac{1}{\sqrt{2}}\mp\frac{1}{2}}$, $y=\sqrt[4]{\frac{1}{2}\pm\frac{1}{\sqrt{2}}}$.] 71. [$x=\pm 3$, $y=\pm 2$.] 72. [$x=\pm 10$, $y=\pm 3$.] 73. [$x=2$, or -1; $y=1$.] 74. [$x=\pm 2$, $y=\pm 6$.]

75. [$x=4$, or -2; $y=2$, or -4.] 76. [$x=\frac{1}{2}(-1\pm\sqrt{3}$, $y=-1$.]

77. [$x=3$, $y=1$.] 78. [5, 4, 3.] 79. [8, 9, 12.] 80. [1, 4, 27.]

81. [18, 32, 16.] 82. [128, 48, 68.] 83. [1, 2, 4.]

84. [$\frac{12}{7}$, $\frac{12}{5}$, -12.] 85. [20, 10, 15.] 86. [4, 5, 6.]

87. [20, 8, 3.] 88. [4, 3, 2.] 89. [3, 2, 5.] 90. [6, 7, 2.]

91. [10, 20, 20.] 92. [45, 30, 30.] 93. [$4\frac{4}{5}$, 8, 24.]

94. [6, 12, 8.] 95. [6, 9, $\frac{1}{2}$.] 96. [$\frac{2abc}{ac+bc-ab}$, $\frac{2abc}{ab+bc-ac}$, $\frac{2abc}{ab+ac-bc}$.] 97. [2, 1, 0.] 98. [$\frac{1}{(b-c)(a-c)}$, $\frac{-1}{(b-c)(a-b)}$, $\frac{1}{(a-c)(a-b)}$.] 99. [24, 60, 120.] 100. [$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.]

101. [1, 2, 3.] 102. [3, 5, 7.] 103. [$(bc^{10}a^{-8})^{\frac{1}{27}}$, $(ab^{10}c^{-8})^{\frac{1}{27}}$, $(ca^{10}b^{-8})^{\frac{1}{27}}$.] 104. [1, 3, 4.] 105. [$\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, 0.] 106. [100, 60, -13, -60.] 107. [3, 7, 11, 20.] 108. [2, 4, 3, 3, 1.]

109. [abc, $ab+ac+bc$, $a+b+c$.] 110. [$\frac{a}{2}$, $\frac{b}{2}$, $\frac{c}{2}$, or $-a$, $2b$, $\frac{c}{2}$.] 111. [m, m, m.] 112. [4, 3, 2.] 113. [2, 3, 6.]

114. [5, 3, 2, and 2, 3, 5.] 115. [± 1, ± 3, ± 5.] 116. [9, 3, 1.]

117. [2, 2, 1, or ± 1, ± 1, 4.] 118. [4, 6, 3.] 119. [3, 2, 1.]

120. [8, 4, 2.] 121. [9, 3, 4.] 122. [$\frac{3}{2}$, 1, $\frac{2}{3}$.] 123. [1, 2, 4, or 9, -6, 4.] 124. [$\frac{bcm}{(a-b)(a-c)}$, $\frac{acm}{(a-b)(c-b)}$, $\frac{abm}{(a-c)(b-c)}$.]

125. $\left[\frac{3abc}{2ab+2ac-bc}, \frac{3abc}{2ab+2bc-ac}, \frac{3abc}{2ac+2bc-ab}.\right]$

126. $\left[\frac{b^2+c^2-a^2}{2bc}, \frac{a^2+c^2-b^2}{2ac}, \frac{a^2+b^2-c^2}{2ab}.\right]$ 127. $\left[\frac{2abc}{ac+bc-ab}, \frac{2abc}{ab+bc-ac}, \frac{2abc}{ab+ac-bc}.\right]$ 128. $\left[\sqrt{\frac{(a^2+b^2)(b^2+c^2)}{2(b^2+c^2)}}, \sqrt{\frac{(a^2+b^2)(b^2+c^2)}{2(a^2+c^2)}}, \sqrt{\frac{(a^2+c^2)(b^2+c^2)}{2(a^2+b^2)}}.\right]$ 129. $\left[\frac{\pm a^2}{a^2+b^2+c^2}, \frac{\pm b^2}{a^2+b^2+c^2}, \frac{\pm c^2}{a^2+b^2+c^2}.\right]$ 130. [0, a, 0.] 131. [1, $1\frac{1}{2}$, 2.]

132. $\left[\left(\frac{b^2c^2}{a}\right)^{\frac{1}{3}}, \left(\frac{a^2c^2}{b}\right)^{\frac{1}{3}}, \left(\frac{a^2b^2}{c}\right)^{\frac{1}{3}}.\right]$ 133. [81, 16, or 16, 81.]

134. [9, 1.] 135. [$x = 1$, or $\frac{1}{5}$; $y = -1$, or $\frac{3}{5}$.] 136. [$x = (\sqrt{a} \pm \sqrt{b})^2$, $y = (\sqrt{b} \mp \sqrt{a})^2$.] 137. $\left[x = \sqrt{\frac{8 \pm 2\sqrt{3}}{13}}, y = \sqrt{\frac{11 \pm 6\sqrt{3}}{13}}\right]$ 138. $\left[\frac{4a}{5}, \frac{5a}{4}.\right]$ 139. $\left[\frac{(1+a^2)^{\frac{2}{3}}}{2\sqrt[3]{2}} \times \left\{1 + \left(\frac{2a}{1+a^2}\right)^2\right\}, a\sqrt[3]{\frac{2a}{1+a^2}}.\right]$ 140. [2, 1.] 141. [625, 1; or 1, 3125.] 142. [5, 4; or $2\sqrt{\frac{2}{5}}$, 10.] 143. [64, 8; or 8, 64.]

144. [8, 1; or 1, 8.] 145. [7, 8.] 146. [$\frac{3}{2}$, $\frac{7}{2}$.] 147. [$7a$, a.]

148 [8, 32.] 149. [5, 1; or -1, -5.] 150. [$\frac{625}{16}$, 25.]

151. $\left[\frac{1}{2}\left(a \pm \sqrt{-3a^2 \mp \frac{1}{2}\sqrt{2(a^4+b^4)}}\right), \frac{1}{2}\left(a \mp \sqrt{-3a^2 \mp \frac{1}{2}\sqrt{2(a^4+b^4)}}\right).\right]$

152. [3, 2; or -2, -3.] 153. [4, 2; or 2, 4.] 154. [± 8, ± 27.]

155. $\left[\pm 2, \pm 1; \text{ or } \pm\left(\frac{3}{2}\right)^{\frac{1}{2}} \quad \pm 2^{-\frac{1}{2}}.\right]$ 156. [4, 2; or 2, 4.]

157. [3, 4; or 4, 3.] 158. [4, 2; or 2, 4.] 159. [4, 3; or 3, 4.]

160. $\left[\pm\frac{c(m+n)}{\sqrt{2(m^2+n^2)}}, \pm\frac{c(m-n)}{\sqrt{2(m^2+n^2)}}.\right]$ 161. $\left[\frac{b}{\sqrt[3]{4a}}, (4a)^{\frac{1}{6}} \times (b-a)^{\frac{1}{2}}.\right]$ 162. [6, 3.] 163. $\left[\frac{ab}{b^2-a^2}(b \pm \sqrt{2a^2-b^2}), \frac{ab}{b^2-a^2}(b \mp \sqrt{2a^2-b^2}).\right]$ 164. $\left[\frac{m \pm \sqrt{2n^2-m^2}}{m^2-n^2}, \frac{m \mp \sqrt{2n^2-m^2}}{m^2-n^2}.\right]$

165. $[-1, \frac{5}{13}, \text{ or } \frac{241}{169}, \frac{5}{13}.]$ 166. $[\frac{-11 \pm \sqrt{153}}{16} . a, \frac{13 \mp \sqrt{153}}{8} . a.]$

167. $[4, -2; \text{ or } 2, -4.]$ 168. $[6, 12; \text{ or } -\frac{8}{3}, -9.]$ 169. $[4, \frac{1}{4}.]$

170. $[4, 3.]$ 171. $[\frac{4a}{3}, \frac{5a}{3}; \text{ or } \frac{3a}{4}, -\frac{5a}{4}.]$ 172. $[1, 4; \text{ or } -1 \pm 2\sqrt{-2}, -2.]$ 173. $[\pm \frac{1}{2b}(a^2+b^2 \pm \sqrt{4a^2b^2-(b^2-a^2)^2}), \pm \frac{1}{2b}(a^2+b^2 \mp \sqrt{4a^2b^2-(b^2-a^2)^2}).]$ 174. $[3, 2; \text{ or } 2, 3.]$ 175. $[3, 2.]$

176. $[3 \pm 2\sqrt{2}, 1.]$ 177. $[8, 2; \text{ or } -12, -\frac{4}{3}.]$ 178. $[\frac{1}{4}(1 \pm \sqrt{2}), \pm\frac{1}{2}.]$ 179. $[\pm 24, 18; \text{ or } \pm 8, 6.]$ 180. $[4, \frac{1}{4}.]$ 181. $[1, 1; \text{ or } -1, -1; \text{ or } \frac{a^2 + a\sqrt{a^2+4}}{2}, \pm\frac{a^2-1}{a^2+1}.]$ 182. $[(\frac{1}{2}a^n + \frac{1}{2}\sqrt{a^{2n}-4b^{2n}})^{\frac{1}{n}}, (\frac{1}{2}a^n - \frac{1}{2}\sqrt{a^{2n}-4b^{2n}})^{\frac{1}{n}}.]$ 183. $[6, 0; \text{ or } 4, 2; \text{ or } 0, -2.]$ 184. $[20, 16.]$ 185. $[2, 1.]$ 186. $[4, 2; \text{ or } 2, 4.]$

187. $[1, 4; \text{ or } 2, \frac{100}{9}.]$ 188. $[1, \pm\frac{1}{3}, -\frac{9}{20}, \pm\frac{1}{3}\sqrt{-1}.]$

189. $[9, 4; \text{ or } 4, 9.]$ 190. $[16, 4; \text{ or } \frac{100}{81}, \text{ and } \frac{64}{81}.]$

191. $[\pm 1, \pm\frac{3}{2}; \text{ or } \pm\sqrt{10}, 3\sqrt{-\frac{1}{2}}.]$ 192. $[\pm 5, \pm 4; \text{ or } \pm\frac{10}{\sqrt{3}}, \pm\frac{8}{\sqrt{3}}.]$ 193. $[4, 2; \text{ or } \frac{1}{4}, -1; \text{ or } 1 \mp 3\sqrt{-7}, \frac{-7 \pm \sqrt{-7}}{2}.]$ 194. $[8, 2; \text{ or } -12, -\frac{4}{3}.]$ 195. $[2, 2, \frac{1}{4}, 16.]$

196. $[5, 4.]$ 197. $[8, 16; \text{ or } 16, 64.]$ 198. $[1, 1.]$

199. $[25, 9.]$ 200. $[5, 3; \text{ or } -\frac{5}{2}(1 \pm \sqrt{-3}); -\frac{3}{2}(1 \pm \sqrt{-3}).]$

201. $[625, 16.]$ 202. $[5, 4.]$ 203. $[\frac{a^2}{2} + \frac{(b^4+a^2)^2}{16b^4}. \frac{a}{4}\left(b^2 + \frac{a^2}{b^2}\right).]$ 204. $[6\frac{1}{2}, -2\frac{1}{2}; \text{ or } 14\frac{5}{18}, -14\frac{1}{6}.]$ 205. $[4, 3.]$

206. $[\sqrt[n]{a^{-1}c}, \sqrt[n]{b^{-1}d}.]$ 207. $[\pm 8, \pm 27; \text{ or } \pm 8\sqrt{-\frac{12}{5}}, \pm 27\sqrt{\frac{12}{5}}.]$ 208. $[x = -\frac{5}{3}, -\frac{8}{5}, \frac{8}{17}(-1+4\sqrt{-1}); y = -1, \frac{5}{17}(-1+4\sqrt{-1}).]$ 209. $[x = 3 \pm 2\sqrt{2}; y = 1, \text{ or } 17 - 12\sqrt{2}.]$

210. $[\frac{1}{2}b \pm \sqrt[m]{\frac{a}{2} \pm \sqrt{a^2+4}}, \frac{1}{2}(b \pm \sqrt[m]{\frac{a}{2} \pm \sqrt{a^2+4}}.]$

211. [$\left\{b^n(\sqrt{a^{m-n}} \pm \sqrt{a^{m-n}-b^{m-n}})\right\}^{\frac{2}{m+n}}$, $\left\{a^m(b^{n-m} \pm \sqrt{b^{n-m}-a^{n-m}})\right\}^{\frac{2}{m+n}}$.] 212. [$c\,(a^m b^m)^{\frac{1}{m+n}}$; $(a^m b^n)^{\frac{1}{m+n}}$.]

213. [1, 4; or $-1 \pm 2\sqrt{-2}$, -2.] 214. [8, 4; or -4, 1; or $84 \mp 16\sqrt{6}$, $(-4 \pm 2\sqrt{6})^2$.] 215. [$\frac{a}{2} \pm \sqrt{\left\{-\frac{3a^2}{4} \mp \sqrt{\frac{d^4+a^4}{2}}\right\}}$; $\frac{a}{2} \mp \sqrt{\left\{-\frac{3a^2}{4} \mp \sqrt{\frac{d^4+a^4}{2}}\right\}}$.] 216. [144, 9; 9, 144.] 217. [5, 3; or $-4\frac{3}{5}$, $-2\frac{188}{250}$.] 218. [$\frac{-11 \pm \sqrt{153}}{16} \times a$; $\frac{13 \pm \sqrt{153}}{16} \times a$.] 219. [$11 \pm 2\sqrt{26}$; $11 \mp 2\sqrt{26}$.] 220. [2, 1.]

221. [$\frac{2m\sqrt{3n^3}}{\sqrt{3n^3} \pm \sqrt{4m^3-n^3}}$; $\frac{2m\sqrt{3n^3}}{\sqrt{3n^3} \mp \sqrt{4m^3-n^3}}$.] 222. [$x = y = \sqrt[3]{m+n}$; or $\frac{\alpha+\beta}{2\sqrt[3]{a}}$, $-\frac{\alpha+\beta}{2\sqrt[3]{a}}$, and $\alpha^2 = 2n \mp \sqrt{m^2-3mn+5n^2}$, $\beta^2 = -2m \mp \sqrt{m^2-3mn+5n^2}$.] 223. [$-\frac{1}{2}(b \mp \sqrt{2a^2-b^2+c^2})$; $\frac{1}{2}(c^2-2a^2 \pm \sqrt{2a^2-b^2+c^2})^{\frac{1}{2}}$.] 224. [$x = \frac{1}{2}(a^2 \pm a\sqrt{a^2+4}) = m$; $y = \frac{m+\sqrt{m}}{1+m-\sqrt{m}}$.] 225. [2, 4; or $\frac{17}{4} + 3\sqrt{2}$, $\frac{17}{2} + 6\sqrt{2}$.]

226. [$\frac{1}{2}(\sqrt{\sqrt[3]{3}+3} + \sqrt{\sqrt[3]{3}-1})$; $\frac{1}{2}(\sqrt[3]{3} \times \sqrt{\sqrt[3]{3}+3} \pm \sqrt{\sqrt[3]{9}-1}$.]

227. [± 4, ± 8.] 228. [$\frac{5}{2}\sqrt{10 \pm 2\sqrt{5}}$; $\frac{5}{2}\sqrt{2 + \frac{2}{\sqrt{5}}}$.]

229. [3, 2.] 230. [3, 1; or ± 3, ± 2; or $\pm 2\sqrt{-1}$, $\pm 3\sqrt{-1}$.] 231. [± 2, ± 1; or $\pm\sqrt{-1}$, $\pm 2\sqrt{-1}$.]

232. [$\pm\frac{5}{2}$, $\pm\frac{1}{2}$, or $\pm\frac{1}{2}$.] 233. [4, $\frac{1}{4}$; or $-2 \pm 2\sqrt{-3}$, $-\frac{13 \pm 3\sqrt{-3}}{8}$.] 234. [4, 12; or $-\frac{4}{17}$, $\frac{36}{19}$.] 235. [2; or -2, 6.] 236. [$\frac{5}{8}(3 \mp 2\sqrt{3} \pm \sqrt{21})$; $\frac{5}{8}(3 \pm 2\sqrt{3} \mp \sqrt{21})$.]

237. [$\frac{1}{4b} \cdot \left\{a^2+b^2 \pm \sqrt{(3a^2-b^2)(3b^2-a^2)}\right\}$; $\frac{1}{4b} \cdot \left\{a^2+b^2 \mp \sqrt{(3a^2-b^2)(3b^2-a^2)}\right.$.] 238. [8, 27.] 239. [8, or -27.]

240. [$\sqrt{\frac{abc}{d}}, \sqrt{\frac{ad}{bc}}, \sqrt{\frac{bd}{ac}}, \sqrt{\frac{cd}{ab}}$.] 241. [4, 9; or $\frac{1}{15}, \frac{2}{15}$.]

242. [$\frac{1}{2} \pm \frac{1}{4}\sqrt{(4a+2 \mp 2\sqrt{8a^2-4a-8b+1})}$; $\pm \frac{1}{4}\sqrt{(4a+2 \pm 2\sqrt{8a^2-4a-8b+1})}$.] 243. [$\mp \frac{\sqrt{2\sqrt{3}-2} \times \sqrt{2\sqrt{3}\pm 2}}{\sqrt{2\sqrt{3}} \mp (\sqrt{3}-1)}$, $\sqrt{\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}}}$.] 244. [$\sqrt{\frac{r^2 \log a}{r^2 \log m + s^2 \log a}}$; $\sqrt{\frac{s^2 \log a}{r^2 \log m + s^2 \log n}}$.]

245. [$\pm \frac{1}{2}(4 \pm \sqrt{6}) \pm \sqrt{18 \pm 8\sqrt{6}}$; $\frac{1}{2}(2 \pm \sqrt{6} \pm \sqrt{18 \pm 8\sqrt{6}}$.]

246. [± 3, 1; or $\pm 3\sqrt[4]{-10}$, -10, $\pm \frac{1}{3}\sqrt{-3\sqrt{10}}$, $\frac{10}{9}$.]

247. [2, 1; or $\frac{1}{10}, \frac{1}{21}$; or $\frac{1}{32}, \frac{1}{28}$.] 248. [$\pm 9$, ± 4; or $\pm \frac{45}{4}$, $\pm \frac{16}{5}$; or $\pm \frac{8}{\sqrt{-271}} \mp \frac{\sqrt{-271}}{8}$.] 249. [$\pm an$, $\pm bn$; or $\frac{1}{2}(a-b+n \pm \sqrt{(a-b+n)^2-4an})$; $-\frac{1}{2}(a-b-n \pm \sqrt{(a-b+n)^2-4an})$.] 250. [$a-b$, or $a-\frac{b}{2}(n \pm \sqrt{n^2-4})$; $\frac{1}{2}(a-\frac{b}{2}(n \pm \sqrt{n^2-4}))$; where $n = \frac{1}{2b}(a-b \pm \sqrt{a^2+2ab+5b^2})$.]

251. [$\pm \frac{\sqrt{5}}{2}$, $\pm \frac{\sqrt{6}}{2}$; or 0, $\pm\sqrt{-1}$.] 252. [$\frac{5}{2}\sqrt{10 \pm 2\sqrt{5}}$; $\frac{5}{2}\sqrt{2 + \frac{2}{\sqrt{5}}}$.] 253. [$\frac{1}{4}\left(\frac{b^2}{2}+\frac{\sqrt[3]{2a^2}}{b^2}\right)^2 + 2\sqrt[3]{2a^2}$, $\frac{b^2}{2}\sqrt{\frac{a}{2}+\frac{a}{b^2}}$.]

254. [2, $1+\frac{1}{4}\sqrt{6}$.] 255. [12, 2; or $-\frac{19}{2}$, $-\frac{19}{12}$.] 256. [8, 2; or -12, $-\frac{4}{3}$.] 257. [$\frac{3}{2}(19 \pm \sqrt{105})$; $\frac{1}{6}(3 \pm \sqrt{105})$.]

258. [$\frac{a}{2}(1 \pm \sqrt{3})$; $a\sqrt{1 \pm \frac{11\sqrt{3}}{18}}$.] 259. [$\frac{c}{2}(1 \pm \sqrt{5})$, $-\frac{c}{2}(1 \mp \sqrt{5})$.] 260. [4, 64; or $\frac{16}{9}$, $\frac{256}{9}$.] 261. [$\frac{1}{8}(b^{\frac{1}{3}}+\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}})^3$; $\frac{1}{8}(b^{\frac{1}{3}}-\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}})^3$.] 262. [2, or $\frac{34}{81}$, $\frac{1446}{961}$.]

263. [36, 2.] 264. [$\pm\frac{8}{27}$, or $\pm\frac{1}{8}$, or $\pm\left(\frac{\sqrt{385} \pm \sqrt{-335}}{12}\right)^3$; $\pm\left(\frac{385 \mp \sqrt{-335}}{12}\right)^3$.]

X.

No. 1. [*Ans.* 30.] 2. [5, 10.] 3. [6.] 4. [25.] 5. [5, 6.] 6. [8, 15.] 7. [C. 800*l.*, B. 1000*l.*, A. 1400*l.*] 8. [224, 288.] 9. [280.] 10. [1½, 1.] 11. [A. 6*l.* 10*s.*, B. 3*l.* 10*s.*] 12. [15 and 25 years.] 13. [15625.] 14. [Numerator $\frac{9}{11}$, denominator $\frac{26}{11}$.] 15. [3, 4, 5.] 16. [36*l.*, 12*l.*, 16*l.*] 17. [Son 2000*l.*, daughter 800*l.*] 18. [5*l*] 19. [60*l.*, 140*l.*, 200*l.*] 20. [2 o'clock.] 21. [92½ miles from London in 10 hours.] 22. [180½ guineas 22 crowns.] 23. [8*l.* 15*s.*] 24. [A. 22 guineas, B. 26 guineas.] 25. [8 hours.] 26. [1000 men.] 27. [A. 120*l.*, B. 30*l.*, C. 60*l.*, D. 10*l.*] 28. [70.] 29. [190, and 185.] 30. [144.] 31. [3*s.* 9*d.*] 32. [32, 8.] 33. [540*l.*] 34. [120.] 35. [$\frac{1-bc}{a-b}$, $\frac{ac-1}{a-b}$.] 36. [A. 550*l.*, B. 650*l.*, C. 850*l.*] 37. [370*l.*] 38. [A's 135 acres, B's 297, C's 432.] 39. [25, 5.] 40. [300 days] 41. [8, 6.] 42. [$\pm\sqrt{\frac{5}{2}}$, $\pm\sqrt{\frac{5}{4}}(1\pm\sqrt{5})$.] 43. [2880 men.] 44. [$\frac{60}{13}$, $\frac{40}{13}$, $\frac{30}{13}$.] 45. [$\frac{12}{11}$, $\frac{21}{11}$, $\frac{32}{11}$, $\frac{45}{11}$.] 46. [$\frac{4}{15}$.] 47. [$\frac{5n-4p}{2}$ and $\frac{4p-3n}{2}$.] 48. [15 gal. brandy, 21 cyder, 24 wine.] 49. [15 gals., 23 gals., 8 gals.] 50. [$\frac{22a-21b}{20a(a-b)}\times$ £P. each.] 51. [24000 men.] 52. [2*s.* 6*d.*] 53. [200*l.*, 300*l.*, 260*l.*] 54. [A. in $a+\sqrt{ab}$ hours, B. in $b+\sqrt{ab}$ hours.] 55. [3000 men.] 56. [15 and 45.] 57. [15, 21, 27.] 58. [2000*l.*] 59. [2, 10.] 60. [40½ guineas, 21 crowns.] 61. [7500*l.*] 62. [29 miles.] 63. [63, 84, 105.] 64. [A's 5*s.*, B's 3*s.*] 65. [72 apples, 60 pears.] 66. [520.] 67. [15 miles, 2 miles per hour] 68. [168.] 69. [18, 11.] 70. [3600*l.*, 2400*l.*, 1800*l.*] 71. [Stream ⅝ mile per hour, time 4 and 6 hours.] 72. [24 horses, 60 black cattle, 165 sheep.] 73. [45*l.*] 74. [1 mile.] 75. [240*l.*, 120 acres.] 76. [45 miles, and 105 miles.] 77. [A. in 24 days, B. in 48 days.] 78. [22 men, 18 women, 50 children.] 79. [20*l.*] 80. [A. 135, B. 297, C. 432.] 81. [Each of equal ones in 32 hours, the other in 24 hours] 82. [30 days' work, 18 idle.] 83. [450 bags.] 84. [12 women, 21 children.] 85. [3 miles and 7 miles.] 86. [7,

15, 48.] **87.** [Course 1080 yards, $16\frac{1}{2}$ minutes.] **88.** [A. 300*l.*, B. 100*l.*] **89.** [16 and 20 tons.] **90.** [8 hours.] **91.** [Spokes in hind wheel 12, in fore 8; radii as 3 : 2.] **92.** [30, 40.] **93.** [24 guineas.] **94.** [36 miles.] **95.** [Expenditure 60*l.*, produce 90*l.* and 120*l.*] **96.** [16.] **97.** [9 bush. of wheat, 12 bush. of rye.] **98.** [189 discharges of the second.] **99.** [3 miles.] **100.** [10 outsides, 18*s.*] **101.** [32 ships.] **102.** [30 and 12.] **103.** [A. in 105 min., B. in 210 min., C. in 420 min., together in 60 min.] **104.** [700*l.* and 100*l.*] **105.** [2, 4, 7, 9.] **106.** [224 lbs.] **107.** [11 o'clock.] **108.** [6, 12, 3, 27.] **109.** [256.] **110.** [5*l.* 5*s.*] **111.** [2 hrs. 24 min.] **112.** [6 miles per hour.] **113.** [A. 1680*l.*, B. 1440*l.*, C. 1280*l.*] **114.** [$38\frac{2}{11}$ minutes after 7 o'clock.] **115.** [7 bushels of barley at 7*s.* per bushel; wheat 11*s.* per bushel.] **116.** [40 bushels at 10*s.* per bushel.] **117.** [A. 11 days at 35 miles per day; B. 6 days at 21 miles.] **118.** [6 hours and 3 hours.] **119.** [8 oz. of first, 5 oz. of second, 3 oz. of third.] **120.** [432 yds.] **121.** [19 days.] **122.** [A. 5, B. 4.] **123.** [In the tack next greater than $\frac{39q-14r}{39p}$.] **124.** [12 days.] **125.** [3200 men.] **126.** [6 miles per hour; 60 miles.] **127.** [At the 25th milestone.] **128.** [420 of copper, 85 of tin.] **129.** [30600*l.*] **130.** [76 miles from A.] **131.** [Debt 1031*l.* 5*s.*; rate per cent. $7\frac{29}{33}$*l.*; price of bonds $78\frac{11}{13}$.] **132.** [Distance 60 miles; B.'s rate 5, and C.'s rate 10 miles per hour.] **133.** [$2a$ and $3a$.]

XI.

No. **1.** [*Ans.* 5, 8.] **2.** [18, 14.] **3.** [6, or -7.] **4.** [43, or 34.] **5.** [25 and 15 yards.] **6.** [3 and 6.] **7.** [$\sqrt{\pm\frac{1}{\sqrt{5}}}$ and $\sqrt{\pm\frac{1}{\sqrt{5}}} \times \frac{3\pm\sqrt{5}}{2}$.] **8.** [24 and 12 yards.] **9.** [3 and 5.] **10.** [1, 16, 81.] **11.** [9 and 6.] **12.** [2 and 4.] **13.** [$\frac{1}{2}(1\pm\sqrt{5})$; $\frac{1}{2}(3\pm\sqrt{5})$.] **14.** [6 and 4.] **15.** [2, 5, and 8.] **16.** [30 and

25 yards.] 17. [4 and 7.] 18. [$\sqrt{ab}$, $\sqrt{ab^{-1}}$.] 19. [3, 5, 7.] 20. [1 and 4.] 21. [85 and 76.] 22. [24, 16, and 18.] 23. [2.] 24. [5 and 8.] 25. [8.] 26. [10 and 6.] 27. [12.] 28 [32.] 29. [15.] 30. [$\frac{10}{11}$, $\frac{1}{11}$.] 31. [120 and 80.] 32. [28, 21.] 33. [Sherry 2*l*., claret 3*l*., per dozen.] 34. [12.] 35. [60 yds.] 36. [12 men, 15 women.] 37. [A. 40, B. 60.] 38. [7.] 39. [50 sheep.] 40. [4, and $-2 \pm \sqrt{-13}$.] 41. [7 and 21 yards.] 42. [138*l*.] 43. [20 shares, at 45*l*. each.] 44. [$\sqrt{\frac{a+b}{2}}$, $\sqrt{\frac{a-b}{2}}$.] 45. [7 trains.] 46. [4550 men.] 47. [3 miles.] 48. [23.] 49. [342, 456.] 50. [64, 48, 36, and 27 gallons.] 51. [A. 36, and B. 30, miles.] 52. [$\frac{m\sqrt{b}}{\sqrt{m^2+n^2}}$, $\frac{n\sqrt{b}}{\sqrt{m^2+n^2}}$.] 53. [24.] 54. [15, 12, 9.] 55. [8 and 7.] 56. [6 and 18.] 57. [432.] 58. [80*l*.] 59. [32 bushels of wheat, 48 bushels of barley.] 60. [6 of wheat, 8 of oats, 10 of barley.] 61. [5, 6, 9.] 62. [A. 10 miles, B. 9 miles, per hour.] 63. [54, 18, 6, 2.] 64. [2, 4, 6.] 65. [21 quartos at 21*s*. each, each octavo 5¼*s*., each folio 4½ guineas.] 66. [5 and 7.] 67. [2, 4, 6.] 68. [25 and 16 yards.] 69. [A.'s 400*l*., B.'s 900*l*., C.'s 144*l*.] 70. [60 apples, 20 pears.] 71. [2, 5, 8.] 72. [$1+\sqrt{-1}$ and $1-\sqrt{-1}$; $\frac{1}{2}(a \pm \sqrt{a^2-4b^2})$ and $\frac{1}{2}(a \mp \sqrt{a^2-4b^2})$.] 73. [25 gallons of brandy, 20 gallons of rum.] 74. [5 feet and 4 feet.] 75. [36 soldiers, 9*l*. each; each sailor 12*l*.] 76. [5*s*., 7*s*., 9*s*.] 77. [108*l*., 144*l*., 192*l*., 256*l*.] 78. [18, 24, 126, 1770.] 79. [3, 6, 12.] 80. [3, 5, 7, 9.] 81. [25, 13, 6.] 82. [2, 4, 8.] 83. [$\frac{22m-21n}{20m(m-n)}p$ £.] 84. [2, 4, 8.] 85. [A. at 4½ o'clock, B. at 5 o'clock.] 86. [23 calves at 24*s*., 24 sheep at 6*s*.] 87. [In 2 and 9 days.] 88. [1, 3, 9, 27.] 89. [10 oxen at 5*l*. each, 30 sheep at 33⅓*s*.] 90. [Side of triangle, 20 yards; of parallelogram, 20 and 12 yards.] 91. [1, 5, 9, 13.] 92. [121, 841.] 93. [70*l*.]

94. [10 gallons.] 95. [3, 2.] 96. [8 days.] 97. [$\frac{3n^2-3n+1}{(n-1)^3}a$£.]
99. [20*l.*] 100. [3 and $3\frac{1}{4}$ miles.] 101. [92.] 102. [5, 11.] 103. [2, 3, 4, 5, 6.] 104. [7, 21, 63.] 105. [4, 7.] 106. [693.] 107. [1200 gallons; 120 gallons hourly leakage.] 108. [40 bushels at 10*s.* per bushel.] 109. [12, 8, $3\frac{1}{3}$ gallons.] 110. [P. 693 votes, Q. 688 votes, R. 736 votes; Plumpers 4, 558, and 3.] 111. [248.] 112. [$\pm\frac{\sqrt{5}}{2}$, $\pm\frac{\sqrt{5}\pm 5}{4}$.] 113. [30 and 25 yards.] 114. [3, 2.] 115. [79 days; 28 men.] 116. [2, 5, 13.] 117. [2, 3, 4, 5, 6 days.] 118. [4 and 5 yards.] 119. [10 persons, the youngest, 12*l.*] 120. [72.] 121. [6 days; 28 men.] 122. [180 miles.] 123. [4, 6, 11.] 124. [144*l.*, 48*l.*, 16*l.*] 125. [48 minutes.] 126. [24.] 127. [5, 4, 3, 2.] 128. [90, 120, 150 yds.] 129. [Born in 1742; age 63 years.] 130. [25*l.*, 16*l.*] 131. [A. went $\frac{mb^2-na^2}{na(m+n)}$, B. went $\frac{b}{ma}\left(\frac{\sqrt{m^2b^2+n^2a^2}}{m+n}\right)$ yds. per min.] 132. [72 and 12.] 133. [40 bushels at 10*s.*, or $12\frac{1}{2}$ bushels at 2*s.* 8*d.*] 134. [A. to B. 10 miles, B. to C. 24 miles, A. to C. 26 miles; rates, 3 and 9 miles per hour.] 135. [A. 12 yards, B. 9 yards, C. 11 yards per stroke; velocities, 24 : 21 : 22.] 136. [A. 3 miles, B. 2 miles.] 137. [27*l.*] 138. [1 to 9 in first mixture, 1 to 4 in second.] 139. [56 miles; A. 7 miles, B. $9\frac{1}{3}$ miles.] 140. [Distance $54\frac{5}{8}$ miles, rates $8\frac{3}{4}$ and 8 miles per hour.] 141. [18*l.* and 32*l.*] 142. [Street, 18 and 30 chains; sewer, 21 chains; distance of mouth from B., 10 chains.] 143. [A.'s and B.'s large warehouses, 52 feet; small warehouses, 20 feet; and C.'s warehouse, 48 feet wide.] 144. [A. 55, B. 30 gall.] 145. [Distance 30 miles, rate of the wherry, 3 and 6 miles per hour; rate of the steamboat, with tide, 12 miles per hour; against tide, 6 miles per hour.] 146. [20 officers, 40 sergeants, 400 privates.] 147. [Time, 2 hours; breadth of each level, 10 feet; the length of the level at the first, 100 feet; at the second, 200 feet.] 148. [Wine : spirits in P. :: 33 : 5; wine : spirits in Q. :: 3 : 35; quantity by A. : quantity by B. :: 4 : 5.]

XII.

No. 2. [*Ans.* $x=7, 14, 21$; $y=2, 7, 12$.] 3. [$x=20$, $y=8$.] 4. [$x=3, 8$; $y=7, 21$.] 5. [48, 19.] 6. [$x=16, 33$; $y=73, 156$.] 7. [$x=7, 2$; $y=1, 4$.] 8. [56, 9.] 9. [9, 7.] 10. [-1, 6.] 11. [5, 2.] 12. [6, 6.] 13. [14, 45.] 14. [$x=91, 82$; $y=10, 20$.] 15. [$x=25, 46$; $y=22, 42$.] 16. [23, 32.] 17. [9, 2.] 18. [3, 2.] 19. [$x=1, 4, 7$; $y=11, 9, 7$.] 20. [$x=15, 2$; $y=3, 11$.] 21. [$x=37, 86, 135$; $y=13, 30, 47$.] 22. [$x=0, 3, 6$; $y=26, 21, 16$.] 23. [$x=10, 15$; $z=6, 10$.] 24. [$x=4$, $y=2$, $z=7$.] 25. [$x=4, 7, 10$; $y=44, 38, 30$; $z=76, 85, 102$.] 26. [$x=1, 2, 3$; $y=5, 3, 1$; $z=3, 4, 5$.] 27. [$x=244, 216$; $y=2, 4$; $z=-25, -21$.] 28. [$x=45, 4, -33$; $y=6, 11, 16$; $z=-4, 15, 34$.] 29. [$x=1, 2, 3$; $y=5, 3, 1$; $z=345$.] 30. [9, 8, 3.] 31. [4, 2, 7.] 32. [$x=1, 3, 5, 21$; $y=11, 3, 2, 1$.] 33. [$x=13$; $y=3$.] 34. [1, 3, 3.] 35. [$x=12, 2$; $y=1, 3$.] 36. [$x=1, 3, 5$; $y=10, 6, 4$.] 37. [1147, 3331.] 38. [7.] 39. [2, infinite.] 40. [$\frac{4}{7}$, $\frac{1}{8}$.] 41. [6.] 42. [6.] 43. [1147.] 44. [7.] 45. [5.] 46. [Men, 4, 15, 26, 37, 48; women, 84, 65, 46, 27, 8.] 47. [3 sheep, 6 lambs.] 48. [Horses, 5, 13, 4; oxen, 29, 56, 83, &c.] 49. [Impossible.] 50. [333, 520, &c.] 51. [17.] 52. [624.] 54. [1, 1, $\frac{3}{5}$, $\frac{6}{5}$. $\frac{2}{5}$, $\frac{6}{5}$, $\frac{16}{15}$, $\frac{8}{15}$, &c.] 55. [3.] 56. [59, 119.] 57. [$(m^2-n^2-2mn)^2$, $(m^2+n^2)^2$, $(m^2-n^2+2mn)^2$.] 58. [6, 28, &c.] 59. [143.] 60. [$\frac{(b^2+n^2)^2}{4n^2}$, $\frac{(b^2-n^2)^2}{4n^2}$.] 61. [7691.] 62. [5, 11, 17.] 63. [104.] 64. [5 and 3, 31 and 20, 57 and 37.] 65. [11 and 7; or 36 and 23.] 66. [1081, 510.] 67. [5, 4, 3; or 104, 81, 66.] 68. [52, 82, 15; 50, 40, 30.] 69. [4128.] 70. [9, 16, and 36, 64, &c.] 71. [25 and 16; 100 and 64, &c.] 72. [2.] 73. [3.] 74. [140.] 75. [4.] 76. [901 men.] 77. [Impossible.] 78. [By giving 6 sovereigns, and receiving 28 dollars.] 79. [3 geese, 15 quails, 2 snipes.] 80. [Of the first, 12, 14, 16; second, 15, 10, 5; third, 3, 6, 9.] 81. [4 oxen, 90 sheep, 1 cow, 5 calves.] 82. [8, 112, 0; or 16, 99, 5.] 83. [12, 2, 6; 11, 5, 4; 10, 8, 2; 9, 11, 0.] 84. [19 oxen; 1 sheep; 80 geese.] 85. [3 horses, 5 cows.]

XIII.

No. 2. [*Ans.* 3·0969.] 3. [2·623; 2·49.] 4. [1·66; 1·2745.] 5. [4·296; 3·069.] 6. [1·537.] 7. [$\frac{\log c - d\log a}{b\log a}$.] 8. [9·5868.] 9. [$\frac{\log c}{m\log a + n\log b}$.] 10. [$\frac{(\log b)^2}{(\log a)^2}$.] 11. [$a^{\frac{1}{3}}b^{-\frac{1}{4}}$.] 12. [31, 62, 10.] 13. [·37166.] 14. [$a^{2n}b^{2m}c^{-2p}$.] 15. [3.] 16. [$x=4$, $y=6$.] 17. [2·342.] 18. [$\frac{nc}{na+b}$, $\frac{c}{na+b}$.] 19. [$\frac{1}{2}\frac{\log(a^2-b^2)}{\log(a+b)}$.] 20. [$a^3 100^{-1}$.] 21. [$\frac{\log a - \log b}{m\log c - n\log b}$.] 22. [$\frac{\log \cdot \frac{1}{2}(1\pm\sqrt{5})}{2\log a}$.] 23. [$\frac{\log(\frac{1}{2}\pm\frac{1}{2}\sqrt{4b+1})}{\log a}$.] 24. [$x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}}$; $y = \left(\frac{p}{q}\right)^{\frac{p}{p-q}}$.] 25. [$\sqrt{\frac{m^2\log c}{m^2\log a + n^2\log b}}$, $\sqrt{\frac{n^2\log c}{m^2\log a + n^2\log b}}$.] 26. [2·25; 3·375.] 27. [3·551; 1·4204.] 28. [1·242.] 29. [4, 33; or -0, 33.] 30. [$x = \frac{1}{2}(4a+1 \pm \sqrt{8a+1})$; $y = \frac{1}{2}(-1 \pm \sqrt{8a+1}$.] 31. [$\frac{\log(1 \pm \sqrt{2}}{2\log a}$.] 32. [$(-\frac{1}{2}\pm\frac{1}{6}\sqrt{57})^8$; $(-\frac{1}{2}\pm\frac{1}{6}\sqrt{57})^4$.] 33. [$\frac{\log(c \pm \sqrt{1+c^2})}{\log a}$.] 34. [2, or -1.] 35. [$\frac{m\log k}{m\log a + n\log b}$; $\frac{n\log k}{m\log a + n\log b}$.] 36. [$x = \pm\frac{m\sqrt{2\log r}}{\sqrt{2m^2\log a + n^2\log b}}$, $y = \pm\frac{n\sqrt{2\log r}}{\sqrt{2m^2\log a + n^2\log b}}$.] 37. [$x = (ab^{-1})^{\frac{b}{a-b}}$, $y = (ab^{-1})^{\frac{a}{a-b}}$.]

XIV.

No. 1. [*Ans.* The latter in each case.] 2. [77 : 80, and 1406 : 1440.] 3. [$3a : 2b$.] 6. [$a+x : a-x$.] 7. [$\frac{a+x}{a}$;

$\frac{a^2-x^2}{a^3-x^3}$.] 9. [$a:b$; $a:b$.] 10. [1 : 1.] 21. [$a:x::x:a-b$, and $x:y::y:2a-x$.] 23. [9 and 15.] 24. [27 and 48.] 25. [6 and 8.] 28. [$s=\frac{1}{2}ft^2$.] 29. [7 : 8.] 30. [$y=7x$.] 31. [$25xy=12(x^2+y^2)$.] 32. [$ay=b\sqrt{a^2-x^2}$.] 33. [$a^2y^2=b^2(x^2-a^2)$.] 34. [$y^2=4ax$.] 38. [2 gall. of first, 14 gall. of second.] 39. [$\sqrt[3]{a^3+a^{13}}$.] 40. [Diamond $\frac{mcx^2}{a^2(1+m)}$; ruby $\frac{cx^{\frac{3}{2}}}{b^{\frac{3}{2}}(1+m)}$.]

XV.

No. 1. [*Ans.* 272.] 2. [100^2, n^2.] 3. [0, $-8\frac{1}{8}$.] 4. [16.] 5. [1.] 6. [$\frac{n-1}{2}$.] 7. [0.] 8. [800.] 9. [$-84\frac{1}{2}$.] 10. [$-637\frac{1}{2}$.] 11. [1, $-4\frac{2}{8}$.] 12. [$\frac{1}{2}$, $-13\frac{3}{4}$.] 13. [$32\frac{6}{7}$, $\frac{2n^2+3n}{7}$.] 14. [$\frac{n}{2(a+b)}(2na-(n+1)b)$.] 15. [0.] 16. [$n((a+x)^2-(n-1)ax)$.] 17. [$\left(\frac{n}{a}-\frac{n}{x}\times\frac{n+1}{2}\right)$.] 18. [3.] 19. [13, or 7.] 20. [4.] 21. [18, or 19.] 22. [4.] 23. [60.] 24. [4, 5, 6.] 25. [-5, 5.] 26. [2, 5, 8.] 27. [6, 8, 10.] 28. [6.] 29. [n^3.] 30. [$\frac{13}{8}$, $\frac{5}{8}$, $\frac{7}{16}$, $\frac{1}{4}$, $\frac{1}{16}$, $-\frac{1}{8}$, $-\frac{5}{16}$.] 31. [-2, -6, -10, -14.] 32. [19.] 33. [8, 10, 12.] 34. [1, 3, 5, &c.] 35. [$\frac{5}{7}$, 1, $\frac{9}{7}$, &c.] 36. [3, 5, 7, 9.] 37. [3, 5, 7, 9.] 38. [$m+n$, $m+n-1$; 0 or 1.] 39. [$\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+$ &c.] 40. [920 yards.] 41. [$\frac{ma-nb}{a-b}(2n+1)$.] 42. [$\frac{7}{2}$, 6, $\frac{17}{2}$.] 43. [$\frac{4}{5}$, $\frac{3}{5}$, &c.; $-\frac{4}{5}$.] 45. [1, 3, 5, &c.] 46. [4, or 7.] 49. [$(q-r)a+(r-p)b+(p-q)c=0$.]

XVI.

No. 1. [*Ans.* $\frac{85}{128}$.] 2. [$\frac{665}{486}$, $\frac{3}{2}$.] 3. [$\frac{1865}{512}$.] 4. [$\frac{939527}{786432}$, $\frac{4}{3}$.] 5. [$\frac{1640}{729}$, $\frac{9}{4}$.] 6. [$\frac{29524}{98415}$, $\frac{3}{10}$.] 7. [$\frac{1-x}{1+x}$.] 8. [$\sqrt{\frac{3}{5}}\times\frac{4651}{3888(\sqrt{6}+\sqrt{5})}$, $\frac{2\sqrt{3}}{\sqrt{30}+5}$.] 9. [$\frac{-\sqrt{2}}{\sqrt{5}+\sqrt{2}}\times\frac{609}{40}$.]

10. $[\frac{7}{4-2\sqrt[4]{8}}.]$ 11. $[\frac{3^{\frac{n}{3}}-1}{3^{n-\frac{4}{3}}(\sqrt[3]{3}-1)}.]$ 12. $[4+3\sqrt{2}.]$

13. $[\frac{3}{5}.]$ 14. $[\frac{25}{9}.]$ 15. $[\frac{1}{x\sqrt{6}\,(3a)^{\frac{n-8}{2}}} \times \frac{(2x)^{\frac{n}{2}}-(3a)^{\frac{n}{2}}}{\sqrt{2x}-\sqrt{3a}}.]$

16. $[\frac{a+x}{b+x}.]$ 17. $[\frac{x^{\frac{3}{2}}}{y^{\frac{1}{2}}(x+y)}.]$ 18. $[x^p \times \frac{x^{nq}-1}{x^q-1}.]$

19. $[\frac{1}{x+y} \times \frac{x^n-(-y)^n}{x^{n-2}}.]$ 20. $[\frac{1}{2}.]$ 21. [4, 8, 16.]

22. [2, 8, 32; 3, 1, $\frac{1}{3}$.] 23. [6, ·1.] 24. [2, 4, 8.]

25. $[2^{n+1}-3,\ 4(2^n-1)-3n,\ 2^n-1,\ 2(2^n-1)-n.]$ 26. [49, 1.]

27. [1, 4, 16, &c.] 28. [1, 2, 4, &c.] 29. [1, 2, 4, 8.]

30. $[\frac{1}{\sqrt[4]{8}},\ \frac{1}{\sqrt{8}},\ \frac{1}{\sqrt[4]{8^3}},$ 100, 40, 16, $\frac{32}{5},\ \frac{64}{25}.]$ 31. $[\frac{1}{3}$, 1, 3, 9, &c.] 35. [117, 351, 1053; 111, 333, 999.] 43. [3, 6, 12, &c.; $13\frac{1}{2}$, $-4\frac{1}{2}$, $1\frac{1}{2}$, −&c. to infinity.] 50. $[\frac{ab}{(n-1)a-(n-2)b}.]$

51. [8, 12; $\frac{6}{5}$, $\frac{4}{3}$, $\frac{6}{11}$, $\frac{3}{7}$, $\frac{6}{17}$, $\frac{3}{10}$.] 52. [1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c.] 54. [$\frac{12}{5}$, 3.]

55. [$\frac{12}{5}$, 3, 4, 6.] 56. $[\frac{(n+1)\,xy}{ny+x},\ \frac{(n+1)\,xy}{(n-1)\,y+2x} \ldots\ldots \frac{xy\,(n+1)}{nx+y}.]$ 57. [−1, 0, 1, $\frac{1}{2}$, $\frac{1}{3}$, &c.; $\frac{7}{44}$, $\frac{7}{34}$, $\frac{7}{24}$, $\frac{1}{2}$, &c.]

58. $[\frac{ab}{2a-b},\ \frac{ab}{3a-2b},\ \frac{ab}{4a-3b} +$ &c.] 60. [104, 234.]

62. $[(p-q)\,ab + (q-r)\,bc + (r-p)\,ac = 0.]$ 63. $[xy^{-1} = 4.]$

XVII.

No. 1. [*Ans.* 252.] 2. [63.] 3. [255.] 4. [14.] 5. [120.] 6. [230300, 18424.] 7. [(i.) 19958400; (ii.) 34650; (iii.) 39916800; (iv.) 3326400.] 9. [$x=15$, $r=6$.] 12. [3628800.] 17. [6.] 18. [$n=83$, $r=42$.] 19. [6.] 20. [63.] 25. [43092000.] 26. [12.]

XVIII.

No. 9. [*Ans.* $-126a^4$.] 10. $[-\frac{5a^{-\frac{35}{2}}b^{18}}{1024}.]$ 11. $[-1400000a^{12}.]$

12. [145.] 13. $\left[\frac{1.6.11....(5r-4)}{5.10.15....(5r)} \cdot \frac{2^r}{(3a)^{r+\frac{1}{3}}}.\right]$ 38. [2^n, 0.]

46. $\left[\frac{1 . 3 . 5....(2n-1)}{1 . 2 . 3....n} 2^n.\right]$

XIX.

No. 34. [*Ans.* 506.] 35. $\left[\frac{n}{2}(6n^2 - 3n - 1).\right]$

36. $\left[\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}\right]$ 37. [440.] 38. $\left[\frac{7n^2}{12}(2n+3) + \frac{n}{12}(3n^3+10).\right]$ 39. $\left[\frac{1+2x-2x^n}{(1-x)^3}.\right]$ 40. [44100.]

41. $\left[\frac{n}{3}(4n^2+3n-1).\right.$ 42. $\left[\frac{3x^n-2x-1}{(x-1)^2}.\right]$ 43. [See 39.]

44. $\left[\frac{8(n-1)x^{n+1}-8nx^n+x^2+bx+1}{(1-x)^3}.\right]$ 45. $\left[\frac{n}{3}(n+1)(n+2).\right]$

46. $\left[\frac{n}{4}(n+1)(n+2)(n+3).\right]$ 47. $\left[\frac{n}{3}(6n^3+20n^2+9n+50).\right]$

48. $\left[\frac{n}{3}(3n^3 + 40n^2 + 84n + 50.\right]$ 49. $\left[\left(\frac{n.(n+1)}{1.2}\right)^2.\right]$

50. $\left[\frac{2n^5}{5} + \frac{3n^4}{4} + \frac{n^3}{6} - \frac{n^2}{4} - \frac{n}{15}.\right]$ 51. [44330.] 52. [8361.]

53. [58780.]

XX.

No. 40. [*Ans.* (i.) Quotients, 1, 1, 3, 5, 4; Convergents, $\frac{1}{1}$, $\frac{3}{4}$, $\frac{5}{7}$, $\frac{67}{88}$; Continned fractions, $\frac{67}{88} = 1 + \frac{1}{3+}\frac{1}{5+}\frac{1}{4}$.

(ii.) Quotients, 1, 1, 1, 1, 2, 3, 8; Convergents, $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{13}{8}$, $\frac{44}{27}$, $\frac{365}{224}$; Continued fractions, $1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{8} = \frac{365}{224}$.

(iii.) Quotients, 3, 1, 1, 11, 1, 1, 4; Convergents, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{7}$, $\frac{23}{81}$, $\frac{25}{88}$, $\frac{48}{169}$, $\frac{217}{764}$; Continued fractions, $\frac{1}{3+}\frac{1}{1+}\frac{1}{1+}\frac{1}{11+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4} = \frac{217}{764}$.

(iv.) Quotients, 2, 1, 2, 2, 1, 3, 2; Convergents, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{7}{19}$, $\frac{10}{27}$, $\frac{37}{100}$, $\frac{84}{227}$; Continued fractions, $\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{2} = \frac{84}{227}$.

(v.) Quotients, 3, 7, 15, 1, 25, 1, 7, 4; Convergents, $\frac{1}{3}$, $\frac{7}{22}$, $\frac{106}{333}$, $\frac{113}{355}$, $\frac{2931}{9208}$, $\frac{3044}{9563}$, $\frac{24239}{76149}$, $\frac{100000}{314159}$; Continued fractions, $\frac{1}{3+}\frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{25+}\frac{1}{1+}\frac{1}{7+}\frac{1}{4} = \frac{100000}{314159}$.]

41. [(i.) Quotients, 3, 22, 1, 4, 2; Convergents, $\frac{1}{3}$, $\frac{22}{67}$, $\frac{23}{70}$, $\frac{114}{347}$,

$\frac{251}{764}$; Continued fractions, $\frac{1}{3+}\frac{1}{22+}\frac{1}{1+}\frac{1}{4+}\frac{1}{2} = \frac{251}{764}$.

(ii.) Quotients, 7, 3, 1, 1, 3; Convergents, $\frac{7}{1}$, $\frac{22}{3}$, $\frac{29}{4}$, $\frac{51}{7}$, $\frac{182}{25}$; Continued fractions, $7+\frac{1}{3+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3} = \frac{182}{25}$.

(iii.) Quotients, 3, 7, 1, 2, 4, 5, 1, 1; Convergents, $\frac{1}{3}$, $\frac{7}{22}$, $\frac{8}{25}$, $\frac{23}{72}$, $\frac{100}{313}$, $\frac{523}{1637}$, $\frac{623}{1950}$, $\frac{1769}{5537}$; Continued fractions, $\frac{1}{3+}\frac{1}{7+}\frac{1}{1+}\frac{1}{2+}\frac{1}{4+}\frac{1}{5+}\frac{1}{1+}\frac{1}{2} = \frac{1769}{5537}$.

(iv.) Quotients, 3, 5, 7, 9; Convergents, $\frac{3}{1}$, $\frac{16}{5}$, $\frac{115}{36}$, $\frac{1051}{329}$; Continued fractions, $3+\frac{1}{5+}\frac{1}{7+}\frac{1}{9} = \frac{1051}{329}$.]

42. [(i.) Quotients, 5, 3, 2, 3, 10, &c.; Convergents, $\frac{5}{1}$, $\frac{16}{3}$, $\frac{37}{7}$, $\frac{127}{24}$, &c.; Continued fractions, $\sqrt{28}=5+\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{10+}$ &c.

(ii.) Quotients, 5, 1, 1, 3, 5, &c.; Convergents, $\frac{5}{1}$, $\frac{6}{1}$, $\frac{11}{2}$, $\frac{39}{7}$, $\frac{206}{37}$, &c.; Continued fractions, $\sqrt{31} = 5+\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{5+}$ &c.

(iii.) Convergents, $\frac{6}{1}$, $\frac{7}{1}$, $\frac{20}{3}$, $\frac{47}{7}$, &c.

(iv.) $\sqrt{50} = 7+\frac{1}{14+}\frac{1}{14+}$ &c.; $\sqrt{17} = 4+\frac{1}{8+}\frac{1}{8+}$ &c.]

XXI.

No. [*Ans.* 882*l.* 2*s.* 6*d.*] 2. [20135*l.* 5*s.* 3*d.*] 3. [36·894 years.] 4. [$6\frac{1}{2}$ years nearly.] 5. [615*l.* 19*s.* 5*d.*] 6. [24*l.* 14*s.* 9·6*d.*] 7. [141*l.* 6*s.* 10*d.*] 8. [A.'s share 2548*l.* 14*s.* 3*d.*, B.'s share 2356*l.* 8*s.* 7*d.*, C.'s share 2094*l.* 17*s.* 2*d.*] 9. [16·67 years nearly.] 10. [$25088\frac{3}{4}$*l.*] 11. [4882*l.* 4*s.* $8\frac{1}{8}$*d.*] 12. [5*l.* 16*s.*] ,13. [7634*l.* 3*s.* $4\frac{4}{5}$*d.*] 14. [6*l.* 16*s.* $6\frac{1}{4}$*d.*] 15. [The freehold worth 214*l.* 2*s.* $9\frac{1}{2}$*d.* more than the leasehold.] 16. [703*l.* nearly.] 17. [35*l.* 8*s.* 5*d.*] 18. [5 per cent.] 19. [2415*l.* 8*s.* $8\frac{1}{2}$*d.*] 20. [241*l.* 9*s.* $2\frac{1}{2}$*d.*] 21. [432*l.* 19*s.*]

XXII.

No. 1. [*Ans.* 5221.] 2. [1465.] 3. [27*t*.] 4. [1024, 249, 252710.] 5. [*e*7*t*8.] 6. [1456.] 7. [35, 61, 2615, 1421.] 8. [2*t*3568, 473184.] 9. [9294, 344, 2704054.] 10. [1295, 216.] 11. [1110111001111, *n*1*e*.] 12. [62*te*, 10787.] 13. [4112.] 14. [122·2.] 15. [1·5462, $1\frac{1}{49}$.] 16. [572640.] 19. [23·2112.] 20. [$3^5+3^4+3^2-3-1$.] 21. [$3^7-3^6-3^5+3^4+3^3-3-1$.] 22. [$3^6+3^5+3^3+3+1$.] 23. [$2{\cdot}5^3+5^2-5+1$.] 29. [$n-1$.]

XXIII.

No. 22. [*Ans.* 40, 7440.] 23. [2^4, 3^3, 5^2, 7, 11; 240; 3690240; 118580.] 24. [525; 4446.]

XXIV.

No. 1. [*Ans.* $x^2 - 7x + 12$; $x^2 - 2x - 15$; $25x^2 - 1$; $9x^2 - 1$. 2. [$x^3 - 1$; $x^5 - 9x^3 + x^2 - 9$.] 3. [$x^4 - 3x^2 - 42x - 40 = 0$; $x^6 - \frac{21x^4}{4} + \frac{21x^2}{4} - 1 = 0$.] 4. [$x^4 - x^3 + \frac{x}{8} - \frac{33}{4} = 0$; $x^3 - 7x^2 + \frac{63x}{4} - \frac{39}{4} = 0$.] 5. [5.] 6. [8.] 7. [1, $\frac{1}{2}(-1 \pm \sqrt{-3})$.] 8. [2, 2, 3.] 9. [6.] 10. [1, 4, 7.] 11. [-1, -10.] 12. [-1, -2.] 13. [1, 3, 5.] 14. [1, 2, 3.] 15. [± 4.] 16. [1, 2, 4.] 17. [1, 3, 9.] 18. [2, 6, 18.] 19. [1, 2, 4, 8.] 20. [3, 3, 2.] 21. [-2, -2, -4.] 22. [2, 3, 6.] 23. [$-\frac{1}{2}$, 1, $\frac{1}{4}$.] 24. [$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$.] 25. [3, 4, 6.] 26. [$y^4 - 25y^3 + 375y^2 - 1260y - 11700 = 0$.] 27. [$y^2 + 2y - 168 = 0$.] 28. [$y^3 - 15y^2 - 24y - 240 = 0$.] 29. [$3y^3 - 10y^2 + 8y - 56 = 0$.] 30. [$y^4 + y^3 - 135y^2 - 808y - 20 = 0$.] 31. [$y^5 - 9y^3 + y^2 - 9 = 0$.] 32. [$y^3 - 16y^2 - 54y + 120 = 0$.] 33. [$y^4 + 10y^3 + 875y - 625$.] 34. [$y^4 + 63y^3 - 108y + 243 = 0$.] 35. [$y^3 + 9y^2 - 36 = 0$.] 36. [$y^4 + 12y^3 + 27y^2 - 68y - 84 = 0$.] 37. [$y^3 - 11y^2 + 38y - 40 = 0$; $y^3 + 7y^2 + 14y + 8 = 0$.] 38. [$y^2 - 11 = 0$.] 39. [$y^2 - 12x - 11 = 0$.] 40. [$2y^4 + 8y^3 - y^2 - 8y - 20$.] 41. [$19y^4 + 206y^3 + 793y^2 + 1232y + 580 = 0$.] 42. [$3y^4 - 9y^3 - 4y^2 - \frac{65y}{9} - \frac{34}{3} = 0$.] 43. [$2x^3 - 15x^2 + 63x - 324 = 0$.] 44. [$y^3 - 8y - 15 = 0$.] 45. [$y^3 - 12y - 26 = 0$.] 46. [$y^3 - y = 0$.] 47. [$y^4 - 14y^2 - 24y + 21 = 0$.] 48. [$y^3 + 6y - 20 = 0$.] 49. [$\pm\sqrt{-1}$; $\frac{-5 \pm \sqrt{21}}{2}$.] 50. [$\pm\sqrt{-1}$, $\frac{1 \pm \sqrt{-3}}{2}$, $\frac{3 \pm \sqrt{5}}{2}$, $2 \pm \sqrt{3}$.] 51. [2, $\frac{1}{2}$, $\pm\sqrt{-1}$.] 52. [1, 1, $\frac{1}{4}(1 \pm \sqrt{-15})$.] 53. [3.] 54. [10.] 55. [8.] 56. [9.] 57. [12.] 58. [1·7836.] 59. [1·692, or 1·35.] 60. [2·529.] 61. [2·49.] 62. [-4, $2 \pm \sqrt{-3}$.] 63. [1, 2, 3.] 64. [4·2643.] 65. [2·858.] 66. [1·02804.] 67. [$-\frac{3}{4}$, $\frac{3 \pm \sqrt{21}}{8}$.]

XXV.

No. 1. [*Ans.* $x^2-4y\sqrt{xy}+3y^2$; $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.]
2. [$\frac{x-y+2\sqrt{xy}}{x^2-xy+y^2}$.] 3. [$ax+2b+\frac{3c}{x}$.] 4. [(i.) $\frac{80}{87}$; (ii.) 2 or $-\frac{3}{7}$; (iii.) $\sqrt{2}$, $1-\sqrt{2}$; (iv.) $\frac{-11\pm\sqrt{153}}{16}.a$; $\frac{-13\pm\sqrt{153}}{8}.a$.]
5. [$4\sqrt{2}$, $6\sqrt{2}$, $8\sqrt{2}$.] 7. [$77760x^4y^2$.] 8. [$n(9-2n)$, $\frac{3}{4}$.]
9. [3, 4.] 10. [907200.] 11. [$\frac{2(x^2+y^2)}{x^2-y^2}$, $\frac{4xy}{x^2-y^2}$, xy.]
12. [$x-y$.] 13. [$x^6+6x^5y^2+15x^4y^3+\ldots.6xy^5+y^6$; 101.]
14. [$3+\sqrt{5}$; $2+\sqrt{-3}$.] 15. [(i.) 13; (ii.) 3, 2; (iii.) 7, or $-6\frac{20}{21}$; (iv.) $\frac{1}{4}(-3\pm\sqrt{657})$, or $-\frac{1}{4}(-3\pm\sqrt{809}$.] 16. [34 miles.]
17. [435; $\frac{5n-1}{4\times5^{n-1}}$; $\frac{5}{4}$.]

XXVI.

No. 1. [*Ans.* $1\frac{1}{3}$.] 2. [12, 20, 30.] 3. [±3, $\pm\sqrt{6}$, 2, -1.]
4. [$\frac{1}{abc}$.] 6. [When $p^2=3q$; $y^3-\frac{p^2-3q}{3}.y-\frac{2p^3}{27}+\frac{pq}{3}-r=0$.]
7. [4, 3, 5; $2\frac{1}{3}$, 3.] 8. [150 miles from place of accident.]
9. [2·7144, $\frac{1}{2}(-3\pm\sqrt{-35}$.]

XXVII.

No. 1. [*Ans.* $ax-a^{\frac{3}{4}}x^{\frac{2}{3}}b^{\frac{3}{5}}y^{\frac{5}{6}}-a^{\frac{1}{4}}x^{\frac{1}{3}}b^{\frac{2}{5}}y^{\frac{1}{6}}+by$; $\sqrt{3}$.] 2. [(i.) $\frac{1}{2}\times(-1\pm\sqrt{3})$; (ii.) $\frac{1}{2}(-3\pm\sqrt{17})$; $\frac{1}{2}(-3\pm\sqrt{22})$; (iii.) 4, 2.]
3. [Distance 60 miles; speed of passenger train 30 miles.] 5. [$1+x+6\left(\frac{x}{3}\right)^2+10\left(\frac{x}{3}\right)^3$ + &c.; $120\left(\frac{x}{3}\right)^{14}$.] 7. [Half-crowns 1, 3, 5, &c. 17; shillings, 41, 36, 31.... 1.] 8. [(i.) 3, or $-\frac{1}{2}$; (ii.) -1, $\frac{1}{4}(5\pm\sqrt{-15})$; (iii.) $\frac{9}{16}$, 0; (iv.) 7, 4.] 9. [$4\frac{1}{2}$.] 11. [11*l*.]

XXVIII.

No. 2. [*Ans.* (i.) $\frac{a}{3}+b$; (ii.) $\frac{a^3}{x(a^2-x^2)}+1$; (iii.) $a^3-3abc+$

b^3+c^3; (iv.) $-6\sqrt{xy}$.] 3. [(i.) $\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)$; (ii.) $\frac{x+1}{2x+5}$.] 4. [(i.) $\frac{1}{2}$; (ii.) $97\frac{1}{2}$; (iii.) 15, 5.] 5. [$\frac{1}{4}$ qt. at 5*s.*, $\frac{3}{4}$ qt. at 3*s.* 6*d.*] 6. [(i.) $1\frac{5}{7}$, $2\frac{2}{5}$, -12; (ii.) 4, 3.] 7. [2225.] 8. [(ii.) ·443783; (iii.) $2^7\times 3^3\times 5^8$; (iv.) 7651.]

XXIX.

No. 2. [*Ans.* (i.) $3(a^2+2b^2)$; (ii.) $\frac{x^3-2x-9}{(x+1)^2\,(x^2-1)}$; (iii.) $x^4+\frac{7x^2y^2}{4}+y^4$; (iv.) x^2+xy+y^2; (v.) $a^2-2b^2+c^2$.] 3. [$x^3-(a+b+c)x^2+(ab+bc+ac)x-abc=0$, $x^3-3ax^2+3a^2x-a^3=0$, $2x-3y$.] 4. [(i.) 13; (ii.) 15, 20.] 5. [1000 men; 800 men.] 6. [(i.) -1, 3, $\frac{1}{8}$; (ii.) 729, 64; (iii.) 0, 0, 0.] 7. [9070.]

XXX.

No. 1. [*Ans.* $\frac{9}{400}$.] 2. [$x^4+\frac{19x^2}{18}+\frac{9}{16}$, $a^3+3a^2+9a+27$.] 3. [$x-3$, $16x^4-1$.] 4. [(i.) $\frac{a}{4x}$; (ii.) $\frac{m}{1-n}$; (iii.) 0; (iv.) $\frac{y}{x+y}$; (v.) $2(a+b+c)$.] 5. [$x^2-x+\frac{1}{4}$, a^2-2u+1.] 6. [(i.) $8\sqrt{2}$; (ii.) $4\sqrt{5}$; (iii.) $\sqrt{19}$, $\sqrt{7}+\sqrt{6}$.] 7. [(i.) 8; (ii.) 2; (iii.) 1, 1; (iv.) 1, -4; (v.) $\frac{4}{3}$, $-\frac{1}{3}$, $\frac{1}{30}(9\pm\sqrt{481})$; (vi.) $\frac{(a-b)^2}{2b}$; (vii.) $\frac{n}{q}$, or $-\frac{p}{m}$; (viii.) 1, $\frac{1}{2}(-3\pm\sqrt{5}$; (ix.) $\frac{1}{4}$, $\frac{1}{5}$.] 8. [n^2, 100, $-21\frac{2}{3}$.] 9. [2^{n-1}, 2^n-1; $(-3)^5$, -182.] 10. [$n=4$.] 11. [20, 16.] 12. [a^6+6a^5x+&c.; $a+8a^{\frac{7}{8}}y^{\frac{1}{8}}+28a^{\frac{3}{4}}b^{\frac{1}{4}}+$&c.] 13. [$\frac{5}{x}-\frac{1}{x^2}-\frac{4}{(x+1)^2}-\frac{5}{x+1}$.] 14. [$1\frac{29}{40}$.] 16. [$a^2-b^2$, $(x-1)^3(x+1)$.] 17. [(i.) $\frac{4}{x+2}$; (ii.) $\frac{8x+9}{(x^2+x+1)\,(x+1)^2}$; (iii.) $\frac{1}{(x+a)\,(x+b)\,(x+c)}$.] 18. [(i.) $9\sqrt{2}$; (ii.) 0.] 19. [$5-\sqrt{3}$, $3(1-\sqrt{-1})$.] 20. [1, 20.] 21. [$a^6+12a^5b+60a^4b^2+$&c.; $\frac{1}{a}\left(1+\frac{x^2}{2a^2}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^4}{a^4}+\text{\&c.}\right)$] 22. [(i.) 7; (ii.) 1, $1\pm 2\sqrt{15}$; (iii.) $\frac{n^2-m^2}{bn-am}$, $\frac{n^2-m^2}{an-bm}$; (iv.) 5, 1;

(v.) 2, 4, 8.] 23. [18 hours.] 24. [$x^4 + px^3 + \frac{p^2x^2}{4} = \frac{r^2x^2}{4s^2} - rx + s^2$; 2, -1.] 25. [$\frac{1}{2(x^2+x+1)} + \frac{1}{2(x^2-x+1)}$.] 26. [$\frac{40}{81} \cdot \frac{b^6}{a^{\frac{22}{3}}} \cdot \sqrt{-1}$, 31570.]

XXXI.

No. 1. [*Ans.* 100 miles.] 5. [(α) 10; (β) 5, 2; (γ) 2, 1.] 6. [160 napoleons = 127 sovereigns; 8 napoleons = 6 sovereigns, + 7*s.*] 7. [48 days.] 8. [(α) $\sqrt{2+\frac{1}{2}\sqrt{-2}} + \sqrt{2-\frac{1}{2}\sqrt{-2}}$; ($\beta$) $\sqrt{2} - \sqrt{3} + 1$.] 11. [145106·1, 44·4.]

XXXII.

No. 1. [*Ans.* $3a^5 - 7a^4x + 8a^3x^2 - 8a^2x^3 + \frac{10ax^4}{3} - x^5$.] 2. [$a + bx + cx^2$.] 3. [(i.) $\frac{x+4}{x^2+2x-2}$; (ii.) $\frac{1}{a} + \frac{1}{x}$; (iii.) $\frac{1-x}{1-2x}$.]

4. [(i.) $5\frac{1}{2}$; (ii.) $\frac{a^2}{b-a}$; (iii.) 9, 7.] 5. [150 yards; A. 30 yards, B. 20 yards per minute.] 6. [$x^2 - ax + 2a^2$, 36.] 7. [(i.) $\sqrt{ba^{-1}}$; (ii.) 3.] 8. [(i.) 4, $-2\frac{2}{3}$; (ii.) $2 \pm \sqrt{17}$; (iii.) $\frac{3}{2}$, $\frac{1}{2}$.] 9. [24, -6, $2a$, $-\frac{a}{2}$.] 10. [(i.) $77\frac{1}{2}$; (ii.) $\frac{865}{648}$; (iii.) $\frac{23}{99}$.] 11. [$3\frac{1}{3}$.]

14. [(i.) $112\frac{1}{2}$; (ii.) $12\frac{1}{2}$; (iii.) $\frac{n}{4}(n+1)a - (n-3)b$.] 16. [3 gall. and 8 gall.] 17. [255.] 18. [10080.] 19. [The senary, 55555, 10000.]

20. [$7 + \cfrac{1}{14 + \cfrac{1}{\sqrt{50}+7}}$.]

21. [1, 2, 3 lbs. at 3*s.* 6*d.*, 7, 4, 1 lb. at 4*s.* 6*d.*, 12, 14, 16 lbs. at 5*s.*] 22. [$1\frac{4}{5}$ days.] 23. [$x = y + \frac{y^3}{3} + \frac{2}{15}y^5 +$ &c.]

24. [2925.] 25. [$3^{\frac{5}{2}} + 9 \times 5^{\frac{2}{3}} + 3^{\frac{3}{2}} \times 5^{\frac{4}{3}} + 3 \times 5^{\frac{6}{3}} + 3^{\frac{1}{2}} \times 5^{\frac{8}{3}} + 5^{\frac{10}{3}}$.]

26. [$(1-2x)^{\frac{5}{2}} = 1 - 5x + \frac{15x^2}{2} - \frac{5x^3}{2} -$ &c.]

Printed by C. F. Hodgson, 1, Gough Square, Fleet Street.